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Licentiate Thesis

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Arithmetic Incidence Functions: A Study of Factorability

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SOPPI, RAUNO: Arithmetic Incidence Functions: A Study of Factorability

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Tiivistelmä

Aritmeettista insidenssifunktiota voidaan lukuteorian termejä käyttämällä luonnehtia funktioksi, jolla on kaikki sekä insidenssifunktiota että kahden muuttujan aritmeettista funktiota määrittelevät ominaisuudet. Tämän luonnehdinnan perusteella funktio $f : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C}$ on aritmeettinen insidenssifunktio, jos $f(x, y) = 0$ aina, kun alkio x ei edellä alkiota y, missä järjestyksen määrittää positiivisten kokonaislukujen tavallinen järjestys.

Jokainen positiivisten kokonaislukujen tavallisen järjestyksen alijärjestys määrittää sitä vastaavan aritmeettisten insidenssifunktioiden aliluokan. Tässä tutkielmassa keskitytään siihen aliluokkaan, jonka määrittää positiivisten kokonaislukujen jaollisuusjärjestys. Ensisijaisena tehtävänä on esittää aritmeettiset insidenssifunktiot yhden muuttujan aritmeettisten funktioiden yleistyksenä, ja yleistää näihin liittyvät multiplikatiivisuuden ja täydellisen multiplikatiivisuuden käsitteet sekä joitakin perustuloksia aritmeettisten insidenssifunktioiden kontekstiin. Toissijaisena tehtävänä on esittää yhteyksiä ja eroja insidenssifunktioita aritmeettisten funktioiden yleistyksenä käsittelevän teorian ja klassisen aritmeettisten funktioiden teorian välillä.

Kaksi konvoluutiota, eli aritmeettisten insidenssifunktioiden D-konvoluutio ja C-konvoluutio, esitellään aritmeettisten funktioiden Dirichlet'n konvoluution ja unitaarikonvoluution yleistyksinä, vastaavassa järjestyksessä. Myös niihin liittyvät Möbiuksen funktiot esitellään.

Pääteemana on aritmeettisten insidenssifunktioiden faktorabiliteetti, termin pitäessä sisällään joukon aiheeseen liittyviä ominaisuuksia. Translaatioinvarianssi ja täydellinen translaatioinvarianssi, jotka sinällään eivät ole varsinaisia faktorabiliteettikäsitteitä, ovat tärkeässä roolissa toimiessaan faktorabiliteetin ja täydellisen faktorabiliteetin osatekijöinä. Semifaktorabiliteetin ja täydellisen semifaktorabiliteetin käsitteet esitetään aritmeettisten funktioiden multiplikatiivisuuden ja täydellisen multiplikatiivisuuden yleistyksinä. Myös nämä kaksi käsitettä ovat osatekijöitä faktorabiliteetille ja täydelliselle faktorabiliteetille, jotka puolestaan esitetään vaihtoehtoisina multiplikatiivisuuden ja täydellisen multiplikatiivisuuden yleistyksinä. Lisäksi esitellään edellä mainittujen faktorabiliteettikäsitteiden duaalikäsitteet.

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Abstract

In terminology of number theory, an arithmetic incidence function can be characterized as a function that possesses all the defining properties of both an incidence function and an arithmetic function of two variables. Under this characterization, a function $f : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C}$ is an arithmetic incidence function if $f(x, y) = 0$ whenever the element x does not preceed the element y, where the order is determined by the standard ordering of positive integers.

Any suborder of the standard ordering of the positive integers determines a subclass of arithmetic incidence functions specific to that suborder. This thesis concentrates on the subclass which is determined by the divisibility ordering of positive integers. The primary objective is to present arithmetic incidence functions as a natural generalization of arithmetic functions of one variable, and to generalize the associated notions of multiplicativity and complete multiplicativity together with some basic results into the context of arithmetic incidence functions. The secondary objective is to present connections and differences between the theory of incidence functions as generalized arithmetic functions and the classical theory of arithmetic functions.

Two types of convolutions, namely the D-convolution and the C-convolution of arithmetic incidence functions, are introduced as generalizations of the Dirichlet convolution and the unitary convolution of arithmetic functions, respectively. Also the related Möbius functions are presented.

The main theme is the factorability of arithmetic incidence functions, where the scope of the term covers a set of related properties. The translation invariance and the complete translation invariance, although not to be regarded as actual concepts of factorability as such, have an important role as elements of factorability and complete factorability. The concepts of semifactorability and complete semifactorability are presented as generalizations of multiplicativity and complete multiplicativity of an arithmetic function. These two concepts are also elements of factorability and complete factorability, which for their part are presented as alternative generalizations of multiplicativity and complete multiplicativity. In addition, the dual concepts for the aforementioned factorability concepts are introduced.

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I also wish to thank the referees, Docent Tapani Matala-aho, Ph.D., from the University of Oulu, and Docent Jukka Pihko, Ph.D., from the University of Helsinki, for their time and effort. I am also very grateful to the referees for their contribution to the final version of this thesis.

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Rauno Soppi

Contents

Chapter 1

Introduction

An arithmetic incidence function is simply a function that possesses all the defining properties of both an incidence function and an arithmetic function of two variables. This characterization of the subject matter at hand reveals that the question is, in fact, about the familiar objects of number theory, namely the arithmetic functions of two variables which have been studied extensively by many researchers during the passed decades. One of the many contexts in which these arithmetic functions have been studied is the theory of incidence functions, where the intention was to develop the combinatorial theory by applying the known results of the theory of arithmetic functions in a more general setting. In this setting the primary focus was not on the arithmetic functions as such but on the more general notion of an incidence function of a partially ordered set.

Theory of incidence functions provides a more general setting, compared to that of arithmetic functions, in which incidence functions can be viewed as generalized arithmetic functions. Within this framework many of the properties of arithmetic functions, or more precisely, the generalizations and analogues of these properties hold in a more or less straightforward fashion. The resulting generalized theory, while necessarily lacking some of the features of the more specific theory of arithmetic functions, also brings forth such features for which there is no exact parallel in the classical theory of arithmetic functions.

The initial motivation for the present author to introduce the term arithmetic incidence function and to investigate the properties of these functions was the hope to gain insights for a deeper understanding of the topic of incidence functions as generalized arithmetic functions. However, during the early phases of the investigation process, the initial objective evolved towards something which can be regarded as a systematic and coherent generalization of some of the central concepts related to arithmetic functions, the framework being that of incidence functions. Despite the change on emphasis, the focus of the present study, guided by the author's initial interest, is on the various generalizations of the notions of multiplicativity and complete multiplicativity of arithmetic functions of one variable. In the center of interest are also the generalizations of the Dirichlet convolution and the unitary convolution of arithmetic functions which are closely associated to the theme of multiplicativity.

Chapter 2 introduces the basic notions and concepts that play signifigant role, more or less explicitly, throughout the rest of the presentation. The most central of these notions are the partially ordered set and the lattice, the latter one essentially being a special case of the first one. Of all types of lattices the distributive lattices, the modular lattices, and the Boolean lattices are introduced explixitly due to their importance in the context of incidence functions. Comprehensive introductions to the partially ordered sets and lattices can be found on the classical textbooks by G. Birkhoff [5] and G. Grätzer [15] (see also [11]). The notion of a local lattice and the related local properties, which are generalizations of the above notions and concepts, have relevance especially in the context of incidence functions (see e.g. [25], [48], [49], [52], and [53]). Chapter 2 includes also a lattice theoretic presentation of the standard ordering of integers and the divisibility ordering of positive integers. Comprehensive introductions to the divisibility of integers can be found on [3], [21], and [33], or any textbook on number theory.

Chapter 3 introduces briefly some of the basics of the theory of arithmetic functions, and it is primarily based on the classical introductions to the subject matter by T. Apostol [3], P. J. McCarthy [25], and R. Sivaramakrishnan [46]. The focus is on the arithmetic functions of one variable and the properties of multiplicativity and complete multiplicativity associated to these functions. Of the many binary operations defined for the arithmetic functions, the Dirichlet convolution and the unitary convolution are given a special attention. The role of chapter 3 is to provide a reference point for the subsequent chapters, and therefore the well-known results are presented without proofs.

Chapter 4 introduces briefly the incidence functions of a partially ordered set as generalized arithmetic functions. This introduction follows the main lines of the development of the associated theory that was initially presented in the end of 1960s by D. A. Smith [47]–[51], and partly reorganized in the beginning of 1970s in [52] into a more concise and coherent presentation. Later, the theory developed by D. A. Smith was restated and supplemented by P. J. McCarthy in his classical introduction to arithmetic functions (see [25, Chapter 7]). As in the case of chapter 3, the role of chapter 4 is to provide a reference point for the subsequent chapters, and therefore the well-known results are presented without proofs. The following historical overview of the theory of incidence functions is owing to both P. J. McCarthy (see [25, pp. 330–332]) and D. A. Smith (see [47]).

The early stages of the theory of incidence functions, also known as the incidence algebra of a locally finite partially ordered set, date back to 1930s and to works of L. Weisner [58], [59] and M. Ward [57]. In the early 1960s, when the incidence algebra of a locally finite partially ordered set had almost already fallen into a state of oblivion, it was revived by G.-C. Rota [34], who proposed it as a tool for the study of combinatorial theory. The implications of G.-C. Rota's groundbreaking approach to combinatorial theory and his work on the subject, especially on the theory of Möbius functions, can be seen in the present day's literature (see e.g. [1], [53]). As D. A. Smith himself commented (see [52, p. 205]), also his interest in this area was stimulated by the works of G.-C. Rota. Moreover, in order to stress G.-C. Rota's overall influence, D. A. Smith's work on the incidence functions as generalized arithmetic functions, in turn, has encouraged also other researchers to engage into more or less comparable investigations (see e.g. [13], [23], [26], [27], [28]).

It should be noted that approximately at the same time when D. A. Smith worked on his theory of generalized arithmetic functions, H. Scheid developed, independently of D. A. Smith's work, a parallel theory of incidence functions of locally finite partially ordered sets (see [37]–[43]), the points of view between these two theories being slightly different. In his reorganized presentation of the subject D. A. Smith (see [52]), while grounding on his own work, also indicates the connections with H. Scheid's work.

Chapters 5, 6, 7, 8, and 9 present a detailed introduction to arithmetic incidence functions and various notions of factorability of these functions. The primary objective of this introduction is to present arithmetic incidence functions as a natural generalization of arithmetic functions of one variable, and to generalize the associated notions of multiplicativity and complete multiplicativity with some related basic results in the context of arithmetic incidence functions. The secondary objective of this presentation, still as important as the former, is to present connections and differences between the theory of arithmetic functions and the theory of incidence functions as generalized arithmetic functions which was the initial motivation of this study.

Chapter 5 presents an introduction of the concept of an arithmetic incidence function, gives a more detailed description of the scope and the objectives of the present study, and introduces the binary operations of addition, multiplication, and convolution of arithmetic incidence functions. In fact, two types of convolution are introduced in detail, namely the D-convolution and the C-convolution which are generalizations of the Dirichlet convolution and the unitary convolution of arithmetic functions, respectively. Also the related Möbius functions are presented.

Chapter 6 presents the notion of translation invariance of an arithmetic incidence function and its stronger counterpart, namely the complete translation invariance. Of these two notions, the translation invariance is familiar from the theory of incidence functions.

Chapter 7 presents the notions of semifactorability and semicompressibility with their stronger counterparts. Of these notions the semifactorability is present also in the theory of incidence functions, although not under that term. The close connection between the notions of semifactorability and semicompresibility rests on the duality of the divisibility ordering of positive integers. As stated above, the semifactorability and semicompressibility are accompanied with stronger versions of these notions, namely the complete semifactorability and the complete semicompressibility, respectively.

The semifactorability and the complete semifactorability are presented as alternative generalizations of the multiplicativity and the complete multiplicativity of an arithmetic function, respectively. For obvious reasons, neither the complete semifactorability nor the complete semicompressibility can be associated with an exact counterpart in the theory of incidence functions.

At this point it is worth noting that many notions associated to arithmetic functions of one variable have also been extended for arithmetic functions of two or more variables. Among these notions are also the multiplicativity and complete multiplicativity of an arithmetic function of several variables introduced by R. Vaidyanathaswamy [56] (see also [46, Chapters 3 and 7]). However, there is a significant difference between these notions and the notions of semifactorability and complete semifactorability. This difference is a result of the definition of an arithmetic incidence function, and therefore the semifactorability and the complete semifactorability offer alternative generalizations of multiplicativity and complete multiplicativity of an arithmetic function of one variable.

Chapter 8 presents the notions of factorability and compressibility with their stronger counterparts. Of these notions the factorability is familiar from the theory of incidence functions. For obvious reasons, neither the factorability nor the compressibility can be associated with an exact counterpart in the theory of arithmetic functions of one variable. The close connection between the notions of factorability and compressibility is eventually realized by the fact that these two properties actually characterize each other, reflecting the duality of the divisibility ordering in its strict sense. As stated above, also the notions of factorability and compressibility are accompanied with stronger versions of these notions, namely the complete factorability and the complete compressibility, respectively.

The factorability and the complete factorability can be regarded as alternative generalizations of the multiplicativity and the complete multiplicativity of an arithmetic function, respectively. Although the complete factorability is familiar from the theory of incidence functions, it turns out that it does not coincide with the complete factorability of an arithmetic incidence function. As in the case of factorability and compressibility, also the complete factorability and the complete compressibility are properties that actually characterize each other.

Lastly, chapter 9 presents some special arithmetic incidence functions and investigates the related factorability properties.

Chapter 2

Ordered Set

2.1 Partially Ordered Set

The notion of an ordered set completes the basic concept of a set, defined as a collection of distinct elements, by formalizing and generalizing the intuitive notion of the order of the elements within a set. The core concept in this formalizing is the *partially ordered set*, where the word 'partially' reflects that, given any two distinct elements of an ordered set, their mutual order need not necessarily be determined.

Definition 2.1. Let P be a nonempty set and $\leq$ a binary relation in the set P. The relation $\leq$ is a *partial ordering* of the set P if

(i) $\forall\, x \in P : x \leq x$ (reflexivity),

(ii) $\forall\, x, y \in P : (x \leq y \text{ and } y \leq x) \Rightarrow x = y$ (antisymmetry),

(iii) $\forall\, x, y, z \in P : (x \leq y \text{ and } y \leq z) \Rightarrow x \leq z$ (transitivity).

The elements $x, y \in P$ are *comparable* if $x \leq y$ or $y \leq x$. Accordingly, the elements $x, y \in P$ are *incomparable* if $x \nleq y$ and $y \nleq x$, where the notation $x \nleq y$ denotes that the ordered pair $\langle x, y \rangle \notin \leq$. The notation $x < y$ denotes that $x \leq y$ and $x \neq y$ (*strict order*), and the notation $x \nless y$ denotes that $x < y$ does not hold.

Definition 2.2. Let $\leq$ be a partial ordering of the set P. The *covering relation* $\prec$ in the set P is defined as follows:

$$\prec \ = \{\, \langle x, y \rangle \mid x < y \text{ and } \forall\, z \in P : (x \leq z \text{ and } z < y) \Rightarrow z = x \,\}.$$

If $x, y \in P$ are such that $x \prec y$, then the element x is an *immediate predecessor* of the element y, which, in turn, is an *immediate successor* of the element x. In other words, the element x *is covered by* the element y and the element y *is a cover* of the element x. As usual, the notation $x \nprec y$ denotes that $x \prec y$ does not hold.

Definition 2.3. The combination of a set $P \neq \emptyset$ and a relation $\leq$ in the set P is a *partially ordered set* (abbrev. *poset*) if the relation $\leq$ is a partial ordering of the set P.

The partially ordered set formed by the set P and the relation $\leq$ is denoted by $(P, \leq)$. The poset $(P, \leq)$ is conventionally referred to as the poset P, in short, if the associated partial ordering $\leq$ is clear from the context.

Definition 2.4. Let $(P, \leq)$ be a poset, and let $x, y \in P$. The set S is a *(closed) interval* of the poset $(P, \leq)$ determined by the elements x and y if

$$\forall\, z \in P : z \in S \Leftrightarrow (x \leq z \text{ and } z \leq y).$$

The notation $[\,x, y\,]$ denotes the interval of the poset $(P, \leq)$ determined by the elements x and y, i.e.

$$[\,x, y\,] = \{\, z \in P \mid x \leq z \text{ and } z \leq y \,\}.$$

Definition 2.5. A poset $(P, \leq)$ is *locally finite* if all of its intervals are finite (i.e. include a finite number of elements).

Remark. The notation $\#S$ denotes the number of elements of a finite set S.

An extreme case of a partially ordered set is a poset in which any given two elements are comparable, thus forming a complete sequence of elements.

Definition 2.6. Let P be a nonempty set (i.e. $P \neq \emptyset$). A partial ordering $\leq$ of the set P is a *total ordering* of the set P if

$$\forall\, x, y \in P : x \leq y \text{ or } y \leq x \quad \text{(comparability)}.$$

Definition 2.7. A poset $(P, \leq)$ is a *totally ordered set (chain)* if the relation $\leq$ is a total ordering of the set P.

Another kind of extreme case of a partially ordered set is a poset in which any given two distinct elements are incomparable.

Definition 2.8. A poset $(P, \leq)$ is an *antichain* if

$$\forall\, x, y \in P : x \leq y \Rightarrow x = y.$$

The detailed investigations of a certain partially ordered set usually focus on the various subsets of the poset, the intervals of the poset being perhaps the most important target.

Definition 2.9. Let $(P, \leq)$ be a poset, and let $S \subseteq P$. The element $s \in S$ is the *least element* of the set S, denoted by $\min S$, if

$$\forall\, x \in S : s \leq x.$$

Definition 2.10. Let $(P, \leq)$ be a poset, and let $S \subseteq P$. The element $s \in S$ is the *greatest element* of the set S, denoted by $\max S$, if

$$\forall\, x \in S : x \leq s.$$

Definition 2.11. Let $(P, \leq)$ be a poset, and let $S \subseteq P$. The element $a \in P$ is the *greatest lower bound (infimum)* of the set S, denoted by $\inf S$, if

(i) $\forall\, x \in S : a \leq x$,

(ii) $\forall\, y \in P : (\forall\, x \in S : y \leq x) \Rightarrow y \leq a$.

Definition 2.12. Let $(P, \leq)$ be a poset, and let $S \subseteq P$. The element $a \in P$ is the *least upper bound (supremum)* of the set S, denoted by $\sup S$, if

(i) $\forall\, x \in S : x \leq a$,

(ii) $\forall\, y \in P : (\forall\, x \in S : x \leq y) \Rightarrow a \leq y$.

From the antisymmetry property of a partially ordered set it follows that the least element of a set, if it happens to exist, is unique. The same applies to the greatest element of a set, the greatest lower bound of a set, and the least upper bound of a set.

Definition 2.13. A poset $(P, \leq)$ is *bounded* if

$$\exists\, x, y \in P : x = \min P \text{ and } y = \max P.$$

Definition 2.14. Let $(P, \leq_1)$ and $(P, \leq_2)$ be posets. The partial ordering $\leq_1$ is a *suborder* of the partial ordering $\leq_2$ and the partial ordering $\leq_2$ is a *refinement* of the partial ordering $\leq_1$ if

$$\forall\, x, y \in P : x \leq_1 y \Rightarrow x \leq_2 y.$$

Specifically, every partial ordering is its own suborder. If the partial ordering $\leq_1$ is a suborder of the partial ordering $\leq_2$ and $\leq_1 \neq \leq_2$, then $\leq_1$ is a *proper suborder* of $\leq_2$.

Definition 2.15. A poset $(Q, \leq_Q)$ is a *subposet* of a poset $(P, \leq_P)$ if

$$\forall\, x, y \in Q : x \leq_Q y \Leftrightarrow x \leq_P y.$$

Since a subposet $(Q, \leq_Q)$ of the partially ordered set $(P, \leq_P)$ inherits its partial ordering $\leq_Q$ from the poset P, it follows by the reflexivity property of a poset that $Q \subseteq P$. Specifically, every nonempty interval of a partially ordered set is a subposet of the poset.

Remark. If $(P, \leq_1)$ and $(P, \leq_2)$ are partially ordered sets and the partial ordering $\leq_1$ is a suborder of the partial ordering $\leq_2$, then the poset $(P, \leq_1)$ is a subposet of the poset $(P, \leq_2)$ if and only if $(P, \leq_1) = (P, \leq_2)$, i.e. $\leq_1 = \leq_2$.

Theorem 2.1. *Every subposet of a totally ordered set (chain) is a totally ordered set (chain).*

2.2 Lattice

The concept of an ordered set builds upon the properties of reflexivity, antisymmetry, and transitivity, which taken together are the elements of the partial ordering. By setting additional requirements for the inner structure of a partially ordered set, one gets the concept of a *lattice*. These additional requirements concern all the individual pairs of elements of a poset, i.e. all its subsets with two elements, and their greatest lower and least upper bounds.

Definition 2.16. A poset $(P, \leq)$ is a *lattice* if

$$\forall\, x, y \in P : \exists\, z, w \in P : z = \inf\{x, y\} \text{ and } w = \sup\{x, y\}.$$

As an immediate consequence of the comparability property of a total ordering (see Definition 2.6) it follows that every totally ordered set (chain) is also a lattice.

Theorem 2.2. *Every totally ordered set (chain) is a lattice.*

Definition 2.17. Let $(L, \leq)$ be a lattice. A set $M \subseteq L$ is a *sublattice* of the lattice L if

$$\forall\, x, y \in M : \exists\, z, w \in M : z = \inf\{x, y\} \text{ and } w = \sup\{x, y\}.$$

The lattice $(L, \leq)$ can be handled as the algebraic structure $(L, \wedge, \vee)$, where the lattice operations *meet* and *join*, denoted by $\wedge$ and $\vee$ respectively, are defined as follows:

$$\begin{aligned} \wedge : L \times L \to L : \wedge(x, y) = \inf\{x, y\}, \\ \vee : L \times L \to L : \vee(x, y) = \sup\{x, y\}. \end{aligned}$$

Theorem 2.3. *Let* $(L, \wedge, \vee)$ *be a lattice. Then*

(i) $\forall\, x \in L : x \wedge x = x$ *and* $x \vee x = x$,

(ii) $\forall\, x, y \in L : x \wedge y = y \wedge x$ *and* $x \vee y = y \vee x$,

(iii) $\forall\, x, y, z \in L : x \wedge (y \wedge z) = (x \wedge y) \wedge z$ *and* $x \vee (y \vee z) = (x \vee y) \vee z$,

(iv) $\forall\, x, y \in L : x \wedge (x \vee y) = x$ *and* $x \vee (x \wedge y) = x$.

Theorem 2.3 states that the lattice operations meet and join satisfy *idempotency laws*, *commutativity*, *associativity*, and *absorption laws*, respectively.

Lattices are classified based on the properties of their inner structures. Of all lattices the most important and widely studied are the *distributive* and *modular* lattices.

Definition 2.18. The lattice $(L, \wedge, \vee)$ is *distributive* if

$$\forall\, x, y, z \in L : x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

Theorem 2.4. *The lattice* $(L, \wedge, \vee)$ *is distributive if and only if*

$$\forall\, x, y, z \in L : x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

Theorem 2.4 presents a characterization of a distributive lattice, which can act as an alternative definition of a distributive lattice.

Theorem 2.5. *Every totally ordered set (chain) is a distributive lattice.*

Theorem 2.6. *A sublattice of a distributive lattice is distributive.*

Definition 2.19. The lattice $(L, \wedge, \vee)$ is *modular* if

$$\forall\, x, y, z \in L : x \leq y \Rightarrow (x \vee z) \wedge y = x \vee (z \wedge y).$$

Theorem 2.7. *If the lattice* $(L, \wedge, \vee)$ *is distributive, then it is modular.*

Theorem 2.7 states, basically, that the distiributive lattices form a subclass of modular lattices.

Theorem 2.8. *A sublattice of a modular lattice is modular.*

The concept of a lattice is generalized by localizing the requirements for the greatest lower and the least upper bounds to apply only within intervals. Furthermore, this generalization suggests the corresponding generalizations of the lattice properties of distributivity and modularity.

Definition 2.20. A poset $(P, \leq)$ is a *local lattice* if every nonempty interval of the poset P is a lattice with respect to the partial ordering that it inherits from the poset P.

Specifically, every lattice is a local lattice.

Definition 2.21. A local lattice $(P, \leq)$ is *locally distributive* if every nonempty interval of the local lattice P is a distributive lattice.

Definition 2.22. A local lattice $(P, \leq)$ is *locally modular* if every nonempty interval of the local lattice P is a modular lattice.

Definition 2.23. Let $(L, \wedge, \vee)$ be a bounded lattice, and let $x \in L$. An element $y \in L$ is the *complement* of the element x if

$$x \wedge y = \min L \text{ and } x \vee y = \max L.$$

Theorem 2.9. *If the lattice* $(L, \wedge, \vee)$ *is bounded and distributive, then its every element has at most one complement.*

Definition 2.24. A bounded lattice $(L, \wedge, \vee)$ is *complemented* if its every element has a complement.

Definition 2.25. A lattice $(L, \wedge, \vee)$ is a *Boolean lattice* (i.e. *Boolean*) if it is complemented and distributive.

Definition 2.26. A local lattice $(P, \leq)$ is *locally Boolean* if every nonempty interval of the local lattice P is a Boolean lattice.

2.3 The Standard Ordering of Integers

Following the usual convention, the set of integers is denoted by $\mathbb{Z}$ and the set of positive integers is denoted by $\mathbb{Z}_+$. The set of nonnegative integers is denoted by $\mathbb{N}$ (i.e. the set of natural numbers).

The *standard ordering* of integers, denoted by $\leq$, refers to the order that is usually attached to integers, i.e.

$$\cdots < -2 < -1 < 0 < 1 < 2 < \cdots ,$$

where $<$ denotes the usual strict order (see Definition 2.1).

Theorem 2.10. *The combination $(\mathbb{Z}, \leq)$, where $\leq$ is the standard order relation in the set $\mathbb{Z}$, is a locally finite totally ordered set (chain).*

Theorem 2.11. *Every subposet of the poset $(\mathbb{Z}, \leq)$, where $\leq$ is the standard order relation in the set $\mathbb{Z}$, is a locally finite totally ordered set (chain).*

Theorem 2.12. *The totally ordered set $(\mathbb{Z}, \leq)$, where $\leq$ is the standard order relation in the set $\mathbb{Z}$, is a locally finite distributive lattice.*

The standard order lattice $(\mathbb{Z}, \leq)$ is, in effect, the same as the algebraic structure $(\mathbb{Z}, \min, \max)$, where the min acts as the lattice operation meet and the max acts as the lattice operation join.

Theorem 2.13. *Every subposet of the lattice $(\mathbb{Z}, \leq)$, where $\leq$ is the standard order relation in the set $\mathbb{Z}$, is a locally finite distributive lattice.*

2.4 The Divisibility Ordering of Positive Integers

Number theory is, in a narrow and the traditional sense, a branch of mathematics devoted mostly to the study of the integers and their relationships. Of all the various properties of integers, especially the notion of divisibility is in the center of interest and extensively studied. Manifesting extremely consistent and well-organized structure, the set of positive integers together with the divisibility relations in this set constitute a text book case of a distributive lattice.

Definition 2.27. An integer $x \in \mathbb{Z}$ is a *factor* (*divisor*) of $y \in \mathbb{Z}$ if

$$\exists\, z \in \mathbb{Z} : y = xz.$$

The notation $x \mid y$ denotes that x is a factor of y. If $x, y \in \mathbb{Z}$ are such that $x \mid y$, then $y = xz$, where $z \in \mathbb{Z}$. Often the associated 'companion factor' of x, in this case the integer z, is denoted by y/x.

Definition 2.28. Let $x, y \in \mathbb{Z}$ be such that at least one of them is unequal to 0. The integer $z \in \mathbb{Z}_+$ is the *greatest common factor* of x and y if

(i) $z \mid x$ and $z \mid y$,

(ii) $\forall\, w \in \mathbb{Z}_+ : (w \mid x \text{ and } w \mid y) \Rightarrow w \leq z$.

Since $1 \in \mathbb{Z}_+$ is a common factor of $x, y \in \mathbb{Z}$, it follows by the well-ordering of integers that the greatest common factor of x and y exist. The uniqueness of the greatest element guarantees the uniqueness of the greatest common factor of x and y, which is denoted by $\operatorname{gcf}(x, y)$.

Theorem 2.14. *Let $x, y \in \mathbb{Z}$. Then*

$$\forall\, w \in \mathbb{Z} : (w \mid x \textit{ and } w \mid y) \Leftrightarrow w \mid \operatorname{gcf}(x, y).$$

Theorem 2.14 states that every common factor of two integers is a factor of their greatest common factor.

Definition 2.29. An integer $y \in \mathbb{Z}$ is a *multiple* of $x \in \mathbb{Z}$ if $x \mid y$.

Definition 2.30. Let $x, y \in \mathbb{Z}$. The integer $z \in \mathbb{Z}_+$ is the *least common multiple* of x and y if

(i) $x \mid z$ and $y \mid z$,

(ii) $\forall\, w \in \mathbb{Z}_+ : (x \mid w \text{ and } y \mid w) \Rightarrow z \leq w$.

Since $|x||y| \in \mathbb{Z}_+$ is a common multiple of $x, y \in \mathbb{Z}$, it follows by the well-ordering of integers that the least common multiple of x and y exist. The uniqueness of the least element guarantees the uniqueness of the least common multiple of x and y, which is denoted by $\operatorname{lcm}(x, y)$.

Theorem 2.15. *Let $x, y \in \mathbb{Z}$. Then*

$$\forall\, w \in \mathbb{Z} : (x \mid w \textit{ and } y \mid w) \Leftrightarrow \operatorname{lcm}(x, y) \mid w.$$

Theorem 2.15 states that the least common multiple of two integers, is a factor of every common multiple of these two integers.

The *divisibility order* of positive integers, i.e. the restriction of divisibility to positive integers, is communicated by the following relation.

Definition 2.31. The *factor relation* $\trianglelefteq$ in the set $\mathbb{Z}_+$ is defined as follows:

$$\trianglelefteq \;\; = \{\, \langle x, y \rangle \mid x, y \in \mathbb{Z}_+ \text{ and } \exists\, z \in \mathbb{Z}_+ : y = xz \,\}.$$

The notation $x \ntrianglelefteq y$ denotes that x is not a factor of y, and the notation $x \triangleleft y$ denotes that x is a proper factor of y, i.e. $x \trianglelefteq y$ and $x \neq y$.

The factor relation $\trianglelefteq$ is a partial ordering of the set $\mathbb{Z}_+$ (see Definition 2.1), and it is a proper suborder of the standard order relation $\leq$ in the set $\mathbb{Z}_+$ (see Definition 2.14).

Definition 2.32. The *covering relation* $\blacktriangleleft$ in the set $\mathbb{Z}_+$ is defined as follows:

$$\blacktriangleleft \;\; = \{\, \langle x, y\rangle \mid x \lhd y \text{ and } \forall\, z \in \mathbb{Z}_+ : (x \unlhd z \text{ and } z \lhd y) \Rightarrow z = x \,\}.$$

If $x, y \in \mathbb{Z}_+$ are such that $x \blacktriangleleft y$, then x is an *immediate factor* of y, which, in turn, is an *immediate multiple* of x.

Theorem 2.16. *The combination* $(\mathbb{Z}_+, \unlhd)$, *where* $\unlhd$ *is the factor relation in the set* $\mathbb{Z}_+$, *is a locally finite lattice.*

The factor lattice $(\mathbb{Z}_+, \unlhd)$ is, in effect, the same as the algebraic structure $(\mathbb{Z}_+, \mathrm{gcf}, \mathrm{lcm})$, where the gcf acts as the lattice operation meet and the lcm acts as the lattice operation join. The factor lattice $(\mathbb{Z}_+, \unlhd)$ can also be referred to as the divisibility order lattice of positive integers.

The idempotency laws, the commutativity property, the associativity property, and the absorption laws, introduced by Theorem 2.3, take the following form, respectively, in the lattice $(\mathbb{Z}_+, \unlhd)$.

Theorem 2.17. *The following properties hold for the lattice operations* gcf *and* lcm *in the lattice* $(\mathbb{Z}_+, \unlhd)$*:*

(i) $\forall\, x \in \mathbb{Z}_+ : \mathrm{gcf}(x, x) = x$ *and* $\mathrm{lcm}(x, x) = x$,

(ii) $\forall\, x, y \in \mathbb{Z}_+ : \mathrm{gcf}(x, y) = \mathrm{gcf}(y, x)$ *and* $\mathrm{lcm}(x, y) = \mathrm{lcm}(y, x)$,

(iii) $\forall\, x, y, z \in \mathbb{Z}_+ : \mathrm{gcf}(x, \mathrm{gcf}(y, z)) = \mathrm{gcf}(\mathrm{gcf}(x, y), z)$
and $\mathrm{lcm}(x, \mathrm{lcm}(y, z)) = \mathrm{lcm}(\mathrm{lcm}(x, y), z)$,

(iv) $\forall\, x, y \in \mathbb{Z}_+ : \mathrm{gcf}(x, \mathrm{lcm}(x, y)) = x$ *and* $\mathrm{lcm}(x, \mathrm{gcf}(x, y)) = x$.

The following two theorems deal with the distributivity and the modularity properties of the factor lattice $(\mathbb{Z}_+, \unlhd)$.

Theorem 2.18. *The lattice* $(\mathbb{Z}_+, \unlhd)$ *is distributive, i.e.*

(i) $\forall\, x, y, z \in \mathbb{Z}_+ : \mathrm{gcf}(x, \mathrm{lcm}(y, z)) = \mathrm{lcm}(\mathrm{gcf}(x, y), \mathrm{gcf}(x, z))$,

(ii) $\forall\, x, y, z \in \mathbb{Z}_+ : \mathrm{lcm}(x, \mathrm{gcf}(y, z)) = \mathrm{gcf}(\mathrm{lcm}(x, y), \mathrm{lcm}(x, z))$.

Remark. Since the lattice $(\mathbb{Z}_+, \unlhd)$ is not complemented, it is not a Boolean lattice. Moreover, it is neither locally Boolean.

Theorem 2.19. *The lattice* $(\mathbb{Z}_+, \unlhd)$ *is modular, i.e.*

$$\forall\, x, y, z \in \mathbb{Z}_+ : x \unlhd y \Rightarrow \mathrm{lcm}(x, \mathrm{gcf}(z, y)) = \mathrm{gcf}(\mathrm{lcm}(x, z), y).$$

Definition 2.33. The *unitary factor relation* $\unlhd|$ in the set $\mathbb{Z}_+$ is defined as follows:

$$\unlhd| \;\; = \{\, \langle x, y\rangle \mid x, y \in \mathbb{Z}_+ \text{ and } \exists\, z \in \mathbb{Z}_+ : y = xz \text{ and } \mathrm{gcf}(x, z) = 1 \,\}.$$

The notation $x \lhd| \; y$ denotes that x is a proper unitary factor of y, i.e. $x \unlhd| \; y$ and $x \neq y$.

The unitary factor relation $\unlhd|$ is a partial ordering of the set $\mathbb{Z}_+$ (see Definition 2.1), and it is a proper suborder of the factor relation $\unlhd$ (see Definition 2.14).

Chapter 3

Arithmetic Functions and Multiplicativity

3.1 Arithmetic Function

In number theory, an *arithmetic function* is a complex-valued, or alternatively a real-valued function defined on the set of positive integers. In other words, the domain of an arithmetic function is the set of positive integers, i.e. the set $\mathbb{Z}_+$, and its codomain is either the set of complex numbers, i.e. the set $\mathbb{C}$, or the set of real numbers, i.e. the set $\mathbb{R}$.

Definition 3.1. A function $f : \mathbb{Z}_+ \to \mathbb{C}$ is an *arithmetic function.*

The set of arithmetic functions is denoted by $\mathbb{A}$.

Remark. An arbitrary field can be set as the codomain of an arithmetic function, but the tradition is to use the more concrete set of complex numbers.

The concept of an arithmetic function is generalized to functions of several variables as follows.

Definition 3.2. A function $f : \mathbb{Z}_+^n \to \mathbb{C}$, where $n \in \mathbb{Z}_+$, is an *arithmetic function of n variables.*

This introduction to arithmetic functions is primarily based on the comprehensive introductory works on the subject matter by T. Apostol [3], P. J. McCarthy [25], and R. Sivaramakrishnan [46], and it concentrates on the arithmetic functions of one variable, the emphasis being on the properties of multiplicativity and complete multiplicativity associated to these functions. For more detailed presentations of the specific aspects of multiplicativity and complete multiplicativity, see e.g. [2], [6], [7], [8], [16], [18], [19], [20], [22], [24], [30], [35], [44], [45], [54], [56], and [60].

During the many decades of the development of the theory of arithmetic functions, several binary operations characterized as convolutions have been defined on the set of arithmetic functions (see e.g. [6], [7], [12], [14], [16], [46], [55], [29]). Following the main theme, this introduction covers only two of these convolutions, also known as products, namely the *Dirichlet convolution* and the *unitary convolution*. Despite their very straightforward nature, also the addition and multiplication of arithmetic functions are introduced briefly.

However, despite the restricted viewpoint, it is worth noting that many notions associated to arithmetic functions of one variable have also been extended for arithmetic functions of several variables. Among these notions is the multiplicativity of an arithmetic function of several variables introduced by R. Vaidyanathaswamy [56] (see also [46, Chapters 3 and 7]).

The term 'arithmetic function' reflects that the main interest is focused on functions possessing a variety of properties that arise from the arithmetic of the underlying domain, namely the set $\mathbb{Z}_+$. The following functions building upon the divisibility of integers are examples demonstrating this aspect of arithmetic functions.

Definition 3.3. The function $\gamma \in \mathbb{A}$ is defined as follows:

$$\gamma : \mathbb{Z}_+ \to \mathbb{C} : \gamma(n) = \prod_{\substack{p \mid n \\ p \in \mathbb{P}}} p,$$

i.e. the value of $\gamma(n)$ is the product of distinct prime factors of n.

Definition 3.4. The function $\omega \in \mathbb{A}$ is defined as follows:

$$\omega : \mathbb{Z}_+ \to \mathbb{C} : \omega(n) = \sum_{\substack{p \mid n \\ p \in \mathbb{P}}} 1,$$

i.e. the value of $\omega(n)$ is the number of distinct prime factors of n.

Definition 3.5. The function $\Omega \in \mathbb{A}$ is defined as follows:

$$\Omega : \mathbb{Z} \to \mathbb{C} : \Omega(n) = \sum_{\substack{p \mid n \\ p \in \mathbb{P}}} n(p), \quad \text{where } n = \prod_{p \in \mathbb{P}} p^{n(p)},$$

i.e. the value of $\Omega(n)$ is the total number of prime factors of n, each counted according to its multiplicity.

Definition 3.6. The function $\theta \in \mathbb{A}$ is defined as follows:

$$\theta : \mathbb{Z}_+ \to \mathbb{C} : \theta(n) = \sum_{\substack{ab = n \\ \mathrm{gcf}(a,b)=1}} 1,$$

where the sum runs over all ordered pairs $\langle a, b \rangle$ of positive integers satisfying $ab = n$ and $\mathrm{gcf}(a, b) = 1$, i.e. the value of $\theta(n)$ is the number of unitary divisors of n.

The function $\theta \in \mathbb{A}$ satisfies the following:

$$\forall\, n \in \mathbb{Z}_+ : \theta(n) = 2^{\omega(n)}.$$

(See [25, p. 36].)

When it comes to the algebra of arithmetic functions, the following three functions play a central role.

Definition 3.7. The *zero function* $\mathit{0} \in \mathbb{A}$ is defined as follows:

$$\mathit{0} : \mathbb{Z}_+ \to \mathbb{C} : \mathit{0}(n) = 0.$$

Definition 3.8. The *zeta function* $\zeta \in \mathbb{A}$ is defined as follows:

$$\zeta : \mathbb{Z}_+ \to \mathbb{C} : \zeta(n) = 1.$$

Definition 3.9. The *delta function* $\delta \in \mathbb{A}$ is defined as follows:

$$\delta : \mathbb{Z}_+ \to \mathbb{C} : \delta(n) = \begin{cases} 1, & \text{if } n = 1; \\ 0, & \text{otherwise.} \end{cases}$$

The arithmetic identity function is defined following the usual convention.

Definition 3.10. The *identity function* $I \in \mathbb{A}$ is defined as follows:

$$I : \mathbb{Z}_+ \to \mathbb{C} : I(n) = n.$$

The family of *power functions* is also an example of arithmetic functions.

Definition 3.11. Let $\alpha \in \mathbb{N}$. The *power function* $\zeta_\alpha \in \mathbb{A}$ is defined as follows:

$$\zeta_\alpha : \mathbb{Z}_+ \to \mathbb{C} : \zeta_\alpha(n) = n^\alpha.$$

Remark. $\zeta_0 = \zeta \in \mathbb{A}$ and $\zeta_1 = I \in \mathbb{A}$.

Definition 3.12. Let $f \in \mathbb{A}$. The *(standard order) summatory function* F of the function f is defined as follows:

$$F : \mathbb{Z}_+ \to \mathbb{C} : F(n) = \sum_{k \le n} f(k).$$

Definition 3.13. Let $f \in \mathbb{A}$. The *divisibility order summatory function* F of the function f is defined as follows:

$$F : \mathbb{Z}_+ \to \mathbb{C} : F(n) = \sum_{d \mid n} f(d).$$

3.2 Addition and Multiplication

The addition and multiplication of arithmetic functions are defined by following the convention that is typical for functions, and therefore they share all the usual properties associated to these operations.

Definition 3.14. Let $f, g \in \mathbb{A}$. A function $h \in \mathbb{A}$ is the *sum* of the functions f and g if

$$\forall\, n \in \mathbb{Z}_+ : h(n) = f(n) + g(n).$$

Definition 3.15. The binary operation $+$ in the set $\mathbb{A}$, i.e. the *(pointwise) addition of arithmetic functions*, is defined as follows:

$$+ : \mathbb{A} \times \mathbb{A} \to \mathbb{A} : +(f, g) = h,$$

where the function h is the sum of the functions f and g, denoted by $f + g$.

Definition 3.16. Let $f, g \in \mathbb{A}$. A function $h \in \mathbb{A}$ is the *product* of the functions f and g if

$$\forall\, n \in \mathbb{Z}_+ : h(n) = f(n)g(n).$$

Definition 3.17. The binary operation $\cdot$ in the set $\mathbb{A}$, i.e. the *(pointwise) multiplication of arithmetic functions*, is defined as follows:

$$\cdot : \mathbb{A} \times \mathbb{A} \to \mathbb{A} : \cdot\,(f, g) = h,$$

where the function h is the product of the functions f and g, denoted by fg.

Theorem 3.1. *The algebraic structure $\langle \mathbb{A}, +, \cdot \rangle$ is a commutative ring with unity.*

The zero function $\mathit{0} \in \mathbb{A}$ is the identity element with respect to the addition of arithmetic functions, and the zeta function $\zeta \in \mathbb{A}$ is the identity element with respect to the multiplication of arithmetic functions.

Theorem 3.2. *A function $f \in \mathbb{A}$ has a multiplicative inverse if and only if*

$$\forall\, n \in \mathbb{Z}_+ : f(n) \neq 0.$$

This multiplicative inverse f^{-1} is defined as follows:

$$f^{-1} : \mathbb{Z}_+ \to \mathbb{C} : f^{-1}(n) = f(n)^{-1}.$$

3.3 Dirichlet Convolution

There are several binary operations, defined on the set of arithmetic functions, that utilize, in one way or another, the arithmetic of the underlying domain, namely the set $\mathbb{Z}_+$. One of the most widely studied of these binary operations is the *Dirichlet convolution*, which builds upon the divisibility of integers. The Dirichlet convolution was introduced by a German mathematician Johann Peter Gustav Lejeune Dirichlet (1805–1859).

Definition 3.18. Let $f, g \in \mathbb{A}$. A function $h \in \mathbb{A}$ is the *Dirichlet convolution* of the functions f and g if

$$\forall\, n \in \mathbb{Z}_+ : h(n) = \sum_{d \mid n} f(d)g(n/d).$$

Definition 3.19. The binary operation $*$ in the set $\mathbb{A}$, i.e. the *Dirichlet convolution of arithmetic functions*, is defined as follows:

$$* : \mathbb{A} \times \mathbb{A} \to \mathbb{A} : *(f, g) = h,$$

where the function h is the Dirichlet convolution of the functions f and g, denoted by $f * g$.

The summation in the Dirichlet convolution of arithmetic functions can be presented also in the following form: Let $f, g \in \mathbb{A}$, and let $n \in \mathbb{Z}_+$. Then

$$(f * g)(n) = \sum_{ab=n} f(a)g(b),$$

where the sum runs over all ordered pairs $\langle a, b \rangle$ of positive integers satisfying $ab = n$.

Theorem 3.3. *Let $f \in \mathbb{A}$. Then $F \in \mathbb{A}$ is the divisibility order summatory function of the function f if and only if*

$$F = f * \zeta, \quad \textit{where } \zeta \in \mathbb{A}.$$

Theorem 3.4. *The algebraic structure $\langle \mathbb{A}, +, * \rangle$ is a commutative ring with unity.*

The zero function $\mathit{0} \in \mathbb{A}$ is the identity element with respect to the addition of arithmetic functions, and the delta function $\delta \in \mathbb{A}$ is the identity element with respect to the Dirichlet convolution of arithmetic functions.

Theorem 3.5. *A function $f \in \mathbb{A}$ has a Dirichlet convolution inverse if and only if $f(1) \neq 0$. This Dirichlet convolution inverse f^{*-1} is defined recursively as follows:*

$$f^{*-1}(n) = \begin{cases} f(1)^{-1} & \textit{if } n = 1, \\ -f(1)^{-1} \sum\limits_{\substack{d \mid n \\ d \neq 1}} f(d) f^{*-1}(n/d) & \textit{otherwise.} \end{cases}$$

Let us define the set $\mathbb{A}_1$ as follows:

$$\mathbb{A}_1 = \{f \mid f \in \mathbb{A} \text{ and } f(1) \neq 0\}.$$

Theorem 3.6. *The algebraic structure $\langle \mathbb{A}_1, * \rangle$ is a commutative group, i.e. an Abelian group.*

Theorem 3.7. *If $h \in \mathbb{A}$ is invertible with respect to the Dirichlet convolution, then*

$$\forall\, f, g \in \mathbb{A} : f = g * h \Leftrightarrow g = f * h^{*-1}.$$

Theorem 3.8. *Let $f \in \mathbb{A}$. If $F \in \mathbb{A}$ is the divisibility order summatory function of the function f, then*

$$f = F * \zeta^{*-1}, \quad \textit{where } \zeta \in \mathbb{A}.$$

3.4 Unitary Convolution

The unitary convolution of arithmetic functions is another binary operation, in addition to the Dirichlet convolution, that builds upon the divisibility of integers. The unitary convolution was introduced by R. Vaidyanathaswamy [56], who called it by the term 'compounding (operation)', and it has been studied extensively e.g. by E. Cohen [9] and [10].

Definition 3.20. Let $f, g \in \mathbb{A}$. A function $h \in \mathbb{A}$ is the *unitary convolution* of the functions f and g if

$$\forall\, n \in \mathbb{Z}_+ : h(n) = \sum_{\substack{d|n \\ \gcd(d,n/d)=1}} f(d)g(n/d).$$

Definition 3.21. The binary operation $*_U$ in the set $\mathbb{A}$, i.e. the *unitary convolution of arithmetic incidence functions*, is defined as follows:

$$*_U : \mathbb{A} \times \mathbb{A} \to \mathbb{A} : *_U (f, g) = h,$$

where the function h is the unitary convolution of the functions f and g, denoted by $f *_U g$.

The summation in the unitary convolution of arithmetic functions can also be presented in the following form: Let $f, g \in \mathbb{A}$, and let $n \in \mathbb{Z}_+$. Then

$$(f *_U g)(n) = \sum_{\substack{ab=n \\ \gcd(a,b)=1}} f(a)g(b),$$

where the sum runs over all ordered pairs $\langle a, b\rangle$ of positive integers satisfying $ab = n$ and $\gcd(a, b) = 1$. Recalling Definition 3.6, let us note that $\theta = \zeta *_U \zeta$.

Theorem 3.9. *The algebraic structure $\langle \mathbb{A}, +, *_U \rangle$ is a commutative ring with unity.*

The zero function $0 \in \mathbb{A}$ is the identity element with respect to the addition of arithmetic functions, and the delta function $\delta \in \mathbb{A}$ is the identity element with respect to the unitary convolution of arithmetic functions.

Theorem 3.10. *A function $f \in \mathbb{A}$ has a unitary convolution inverse if and only if $f(1) \neq 0$. This unitary convolution inverse $f^{*_U -1}$ is defined recursively as follows:*

$$f^{*_U -1}(n) = \begin{cases} f(1)^{-1} & \text{if } n = 1, \\ -\, f(1)^{-1} \displaystyle\sum_{\substack{d|n \\ \gcd(d,n/d)=1 \\ d \neq 1}} f(d) f^{*_U -1}(n/d) & \text{otherwise.} \end{cases}$$

Theorem 3.11. *The algebraic structure $\langle \mathbb{A}_1, *_U \rangle$ is a commutative group, i.e. an Abelian group.*

3.5 Multiplicativity

An arithmetic function is constructed in some orderly fashion if it is partially or completely determined by its values at certain elements of the underlying domain, namely the set $\mathbb{Z}_+$. There is a variety of properties that reflect the above mentioned aspect of arithmetic functions, and one of them is the notion of multiplicativity, which builds upon the divisibility of integers.

Definition 3.22. A function $f \in \mathbb{A}$ is *multiplicative* if

(i) $f(1) = 1$,

(ii) $\forall\, m, n \in \mathbb{Z}_+ : \mathrm{gcf}(m,n) = 1 \Rightarrow f(mn) = f(m)f(n)$.

Remark. Alternatively, a function $f \in \mathbb{A}$ is said to be multiplicative if it is not identically zero, i.e $f \neq \mathit{0}$, and satisfies the condition (ii) of Definition 3.22 (see e.g. [2], [3], [25]). However, this characterization of a multiplicative function is equivalent to Definition 3.22.

Remark. In some contexts a function $f \in \mathbb{A}$ is said to be multiplicative if it satisfies the condition (ii) of Definition 3.22 (see e.g. [22]). Under this characterization also the zero function $\mathit{0} \in \mathbb{A}$ is multiplicative, which is, in fact, the only difference between this characterization and Definition 3.22.

Remark. It is worth noting that, traditionally, a function $f \in \mathbb{A}$ satisfying the condition (ii) of Definition 3.22 is referred to as a *factorable* function (see e.g. [57, p. 357], [20, p. 970]).

For example, the arithmetic functions γ, δ, and ζ_α are multiplicative.

Theorem 3.12. *A function $f \in \mathbb{A}$ is multiplicative if and only if*

(i) $f(1) = 1$,

(ii) $\forall\, n \in \mathbb{Z}_+ : f(n) = \prod_{p\in\mathbb{P}} f(p^{n(p)})$, *where* $n = \prod_{p\in\mathbb{P}} p^{n(p)}$.

Theorem 3.12 states, essentially, that a multiplicative function is completely determined by its values at prime powers.

Theorem 3.13. *A function $f \in \mathbb{A}$ is multiplicative if and only if*

(i) $f(1) = 1$,

(ii) $\forall\, m, n \in \mathbb{Z}_+ : f(\mathrm{gcf}(m,n))f(\mathrm{lcm}(m,n)) = f(m)f(n)$.

The property defined by the condition (ii) of Theorem 3.13 is commonly known as *semimultiplicativity* (see e.g. [19], [31], [32], [46, pp. 237–244]).

Theorem 3.14. *A function $f \in \mathbb{A}$ is multiplicative if and only if*

(i) $f(1) = 1$,

(ii) $\forall\, g, h \in \mathbb{A} : f(g *_U h) = (fg) *_U (fh)$.

Remark. If $f \in \mathbb{A}$ is not identically zero and it fulfills the condition (ii) of Theorem 3.14, then necessarily $f(1) = 1$.

Remark. The result of Theorem 3.14 is presented, e.g., in [54].

Theorem 3.15. *Let $f \in \mathbb{A}$, and let $F \in \mathbb{A}$ be the divisibility order summatory function of the function f. Then f is multiplicative if and only if F is multiplicative.*

The set of multiplicative arithmetic functions is denoted by $\mathbb{A}_M$.

Theorem 3.16. *The algebraic structure $\langle \mathbb{A}_M, * \rangle$ is a commutative group, i.e. an Abelian group.*

3.6 The Möbius Function

The Dirichlet convolution inverse of the zeta function $\zeta \in \mathbb{A}$, known as the Möbius function, is a very important function in the theory of arithmetic functions.

Definition 3.23. The *Möbius function* $\mu \in \mathbb{A}$ is defined as follows:

$$\mu = \zeta^{*-1}.$$

The following result, which can be taken as the defining property of the Möbius function, follows from the fact that $\mu * \zeta = \delta$.

Theorem 3.17. *The delta function $\delta \in \mathbb{A}$ is the divisibility order summatory function of the Möbius function $\mu \in \mathbb{A}$, i.e.*

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.18. *The Möbius function $\mu \in \mathbb{A}$ is multiplicative.*

Theorem 3.19. *The Möbius function $\mu \in \mathbb{A}$ is determined recursively as follows:*

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1, \\ -\sum\limits_{\substack{d|n \\ d \neq n}} \mu(d) & \text{otherwise.} \end{cases}$$

Theorem 3.20. *The following holds for the Möbius function $\mu \in \mathbb{A}$:*

(i) $\forall\, p \in \mathbb{P} : \mu(p) = -1$,

(ii) $\forall\, p \in \mathbb{P} : \forall\, \alpha \in \mathbb{N} : \mu(p^{\alpha+2}) = 0$.

Remark. In some contexts, the Möbius function $\mu : \mathbb{Z}_+ \to \mathbb{C}$ is defined directly, without any reference to the zeta function $\zeta \in \mathbb{A}$, as follows:

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n = p_1 p_2 \cdots p_k, \text{ where } p_1 p_2 \cdots p_k \text{ are distinct primes}, \\ 0 & \text{otherwise.} \end{cases}$$

(See e.g. [36, Chapter 2].)

Theorem 3.21. *The Möbius inversion formula for* $\mathbb{A}$ *is the following:*

$$\forall\, f, g \in \mathbb{A} : f = g * \zeta \Leftrightarrow g = f * \mu.$$

The following theorem presents the Möbius inversion formula, introduced in Theorem 3.21, in a different form.

Theorem 3.22. *The following holds for every* $f, g \in \mathbb{A}$*:*

$$\forall\, n \in \mathbb{Z}_+ : f(n) = \sum_{d|n} g(d) \Leftrightarrow \forall\, n \in \mathbb{Z}_+ : g(n) = \sum_{d|n} f(d)\mu(n/d).$$

3.7 The Euler Function

The *Euler function* ϕ, also known as *Euler's totient function*, is a very important arithmetic function introduced by a Swiss mathematician Leonhard Euler (1707–1783).

Definition 3.24. The *Euler function* $\phi \in \mathbb{A}$ is defined as follows:

$$\phi : \mathbb{Z}_+ \to \mathbb{C} : \phi(n) = \#\{m \mid 1 \le m \le n,\ \mathrm{gcf}(m, n) = 1\}.$$

Remark. A positive integer is a *totative* of $n \in \mathbb{Z}_+$ if it is less than or equal to n and relatively prime to n. The number of totatives of $n \in \mathbb{Z}_+$ is the *totient* of n.

Theorem 3.23. *The power function* $\zeta_1 \in \mathbb{A}$ *is the divisibility order summatory function of the Euler function* $\phi \in \mathbb{A}$*, i.e.*

$$\forall\, n \in \mathbb{Z}_+ : n = \sum_{d|n} \phi(d).$$

Remark. Theorem 3.23 states, essentially, that $\phi * \zeta = I$.

Theorem 3.24. *The Euler function* $\phi \in \mathbb{A}$ *is multiplicative.*

Theorem 3.25. *The Euler function* $\phi \in \mathbb{A}$ *satisfies the following formula:*

$$\phi = \zeta_1 * \mu, \quad \textit{where } \zeta_1, \mu \in \mathbb{A}.$$

Theorem 3.26. *The following holds for the Euler function* $\phi \in \mathbb{A}$*:*

$$\forall\, n \in \mathbb{Z}_+ : n \in \mathbb{P} \Leftrightarrow \phi(n) = n - 1.$$

Theorem 3.27. *The following holds for the Euler function* $\phi \in \mathbb{A}$*:*

$$\forall\, p \in \mathbb{P} : \forall\, \alpha \in \mathbb{N} : \phi(p^{\alpha+1}) = p^{\alpha+1} - p^{\alpha}.$$

Theorem 3.28. *The following holds for the Euler function* $\phi \in \mathbb{A}$*:*

$$\forall\, n \in \mathbb{Z}_+ : \phi(n) = n \prod_{\substack{p|n \\ p \in \mathbb{P}}} (1 - p^{-1}).$$

3.8 Divisor Functions

The family of *divisor functions* is also an example of arithmetic functions utilizing the divisibility of integers, and it has a close connection to the family of power functions.

Definition 3.25. Let $\alpha \in \mathbb{N}$. The *divisor function* $\sigma_\alpha \in \mathbb{A}$ is defined as follows:

$$\sigma_\alpha : \mathbb{Z}_+ \to \mathbb{C} : \sigma_\alpha(n) = \sum_{d|n} d^\alpha,$$

i.e. the value of $\sigma_\alpha(n)$ is the sum of the αth powers of the divisors of n.

The letter τ is used to denote the divisor function $\sigma_0 \in \mathbb{A}$, and therefore

$$\tau : \mathbb{Z}_+ \to \mathbb{C} : \tau(n) = \sum_{d|n} 1,$$

i.e. the value of $\tau(n)$ is the number of positive factors of n. Consequently, the divisor function τ is often referred to as the *number of divisors function*.

The letter σ is used to denote the divisor function $\sigma_1 \in \mathbb{A}$, and therefore

$$\sigma : \mathbb{Z}_+ \to \mathbb{C} : \sigma(n) = \sum_{d|n} d,$$

i.e. the value of $\sigma(n)$ is the sum of positive factors of n. Consequently, the divisor function σ is often referred to as the *sum of divisors function*.

Theorem 3.29. *The divisor functions $\sigma_\alpha \in \mathbb{A}$ satisfy the following formula:*

$$\sigma_\alpha = \zeta_\alpha * \zeta, \quad \textit{where } \zeta_\alpha, \zeta \in \mathbb{A}.$$

For example, $\tau = \zeta * \zeta$ and $\sigma = \zeta_1 * \zeta$.

Theorem 3.30. *The divisor function $\sigma_\alpha \in \mathbb{A}$ is the divisibility order summatory function of the power function $\zeta_\alpha \in \mathbb{A}$.*

Theorem 3.31. *The divisor functions $\sigma_\alpha \in \mathbb{A}$ are multiplicative.*

3.9 Complete Multiplicativity

The notion of complete multiplicativity, as the term suggests, builds upon the concept of multiplicativity by strengthening the requirements. The difference between these two concepts is that, given any two elements of the set $\mathbb{Z}_+$, the complete multiplicativity of a function depends on the function values at these elements, whereas the multiplicativity of a function depends on the function values at these elements only if the elements are relatively prime.

Definition 3.26. A function $f \in \mathbb{A}$ is *completely multiplicative* if

(i) $f(1) = 1$,

(ii) $\forall\, m, n \in \mathbb{Z}_+ : f(mn) = f(m)f(n)$.

Remark. Alternatively, a function $f \in \mathbb{A}$ is said to be completely multiplicative if it is not identically zero, i.e $f \neq \mathit{0}$, and satisfies the condition (ii) of Definition 3.26 (see e.g. [2], [3], [25]). However, this characterization of a completely multiplicative function is equivalent to Definition 3.26.

Remark. In some contexts a function $f \in \mathbb{A}$ is said to be completely multiplicative if it satisfies the condition (ii) of Definition 3.26 (see e.g. [22]). Under this characterization also the zero function $\mathit{0} \in \mathbb{A}$ is completely multiplicative, which is, in fact, the only difference between this characterization and Definition 3.26.

Remark. It is worth noting that, traditionally, a function $f \in \mathbb{A}$ satisfying the condition (ii) of Definition 3.26 is referred to as a *multiplicative* function (see e.g. [57, p. 357], [20, p. 970]).

Remark. In some contexts, the term 'totally multiplicative' is used instead of the term 'completely multiplicative' (see e.g. [60]).

For example, the arithmetic functions δ and ζ_α are completely multiplicative.

Theorem 3.32. *If $f \in \mathbb{A}$ is completely multiplicative, then it is multiplicative.*

The following theorems present some well-known characterizations of a completely multiplicative function.

Theorem 3.33. *Let $f \in \mathbb{A}$ be multiplicative. Then f is completely multiplicative if and only if*

$$\forall\, p \in \mathbb{P} : \forall\, \alpha \in \mathbb{Z}_+ : f(p^\alpha) = f(p)^\alpha.$$

Theorem 3.34. *Let $f \in \mathbb{A}$ be multiplicative. Then f is completely multiplicative if and only if*

$$\forall\, p \in \mathbb{P} : \forall\, \alpha \in \mathbb{Z}_+ : f(p^\alpha) = f(p)f(p)^{\alpha-1}.$$

Theorem 3.35. *A function $f \in \mathbb{A}$ is completely multiplicative if and only if*

(i) $f(1) = 1$,

(ii) $\forall\, n \in \mathbb{Z}_+ : f(n) = \prod_{p \in \mathbb{P}} f(p)^{n(p)}$, *where* $n = \prod_{p \in \mathbb{P}} p^{n(p)}$.

Theorem 3.35 states, esentially, that a completely multiplicative function is completely determined by its values at primes.

Theorem 3.36. *A function $f \in \mathbb{A}$ is completely multiplicative if and only if*

(i) $f(1) = 1$,

(ii) $\forall\, g, h \in \mathbb{A} : f(g * h) = (fg) * (fh)$.

Remark. If $f \in \mathbb{A}$ is not identically zero and it fulfills the condition (ii) of Theorem 3.36, then necessarily $f(1) = 1$.

Remark. The result of Theorem 3.36 was first introduced by J. Lambek [20].

Definition 3.27. Let $g, h \in \mathbb{A}$. The Dirichlet convolution of g and h, that is $g * h$, is *discriminative* if

$$\forall\, n \in \mathbb{Z}_+ : (g * h)(n) = g(1)h(n) + g(n)h(1) \Rightarrow n \in \mathbb{P}.$$

Remark. A property of primes is that

$$\forall\, g, h \in \mathbb{A} : \forall\, n \in \mathbb{Z}_+ : n \in \mathbb{P} \Rightarrow (g * h)(n) = g(1)h(n) + g(n)h(1).$$

For example, $\phi * \zeta\ (= I)$ and $\zeta * \zeta\ (= \tau)$, where $\phi, \zeta, I, \tau \in \mathbb{A}$, are discriminative.

Definition 3.28. Let $g, h \in \mathbb{A}$. The Dirichlet convolution of g and h, that is $g * h$, is *semidiscriminative* if

$$\forall\, n \in \mathbb{Z}_+ : (g * h)(n) = g(1)h(n) + g(n)h(1) \Rightarrow (n = 1 \text{ or } n \in \mathbb{P}).$$

Definition 3.29. Let $g, h \in \mathbb{A}$. The Dirichlet convolution of g and h, that is $g * h$, is *partially discriminative* if

$$\forall\, p \in \mathbb{P} : \forall\, k \in \mathbb{Z}_+ : (g * h)(p^k) = g(1)h(p^k) + g(p^k)h(1) \Rightarrow k = 1.$$

For example, $\zeta * \mu\ (= \delta)$, where $\zeta, \mu, \delta \in \mathbb{A}$, is partially discriminative.

Remark. If $g * h$ is discriminative, then it is also semidiscriminative, and if $g * h$ is semidiscriminative, then it is also partially discriminative.

Remark. If $g * h$ is discriminative, then by the commutativity of the Dirichlet convolution also $h * g$ is discriminative. The same applies naturally to partial discriminativity and semidiscriminativity.

Theorem 3.37. *A function $f \in \mathbb{A}$ is completely multiplicative if and only if*

(i) $f(1) \neq 0$,

(ii) $\exists\, g, h \in \mathbb{A} : f(g * h) = (fg) * (fh)$, *where $g * h$ is discriminative.*

Remark. If conditions (i) and (ii) of Theorem 3.37 hold for $f \in \mathbb{A}$, then necessarily $f(1) = 1$ (see [22, p. 412]).

Theorem 3.38. *A function $f \in \mathbb{A}$ is completely multiplicative if and only if*

(i) $f(1) = 1$,

(ii) $\exists\, g, h \in \mathbb{A} : f(g * h) = (fg) * (fh)$, *where $g * h$ is semidiscriminative.*

Theorem 3.39. *A function $f \in \mathbb{A}$ is completely multiplicative if and only if*

(i) $f(1) = 1$,

(ii) $fI = (f\phi) * f$.

Remark. The result of Theorem 3.39 is presented with a proof, e.g., in [45].

Theorem 3.40. *A function $f \in \mathbb{A}$ is completely multiplicative if and only if*

(i) $f(1) = 1$,

(ii) $f\tau = f * f$.

Remark. The result of Theorem 3.40 is presented with a proof, e.g., in [8].

Theorem 3.41. *A function $f \in \mathbb{A}$ is completely multiplicative if and only if*

(i) $f(1) = 1$,

(ii) $\forall\, g \in \mathbb{A} : f(g * g) = (fg) * (fg)$.

Remark. The result of Theorem 3.41 follows by Theorems 3.36 and 3.40, and it is presented explicitly, e.g., in [44].

Theorem 3.42. *Let $f \in \mathbb{A}$ be multiplicative. Then f is completely multiplicative if and only if*

$$\exists\, g, h \in \mathbb{A} : f(g * h) = (fg) * (fh),$$

where $g * h$ is partially discriminative.

Theorem 3.43. *Let $f \in \mathbb{A}$ be multiplicative. Then f is completely multiplicative if and only if*

$$f^{*-1} = f\mu.$$

Theorem 3.44. *Let $f \in \mathbb{A}$ be multiplicative. Then f is completely multiplicative if and only if*

$$\forall\, g \in \mathbb{A} : g(1) \neq 0 \Rightarrow (fg)^{*-1} = fg^{*-1}.$$

Theorem 3.45. *Let $f \in \mathbb{A}$ be multiplicative. Then f is completely multiplicative if and only if*

$$\forall\, p \in \mathbb{P} : \forall\, \alpha \in \mathbb{N} : f^{*-1}(p^{\alpha+2}) = 0.$$

Remark. The notions of discriminative and partially discriminative Dirichlet convolutions (i.e. products) and the results of Theorems 3.37 and 3.42 were first introduced by E. Langford [22], whereas the notion of semidiscriminativity and the result of Theorem 3.38 were subsequently introduced by P. Haukkanen [18] (see also [30], [35], [46, pp. 39–42]).

Chapter 4

Incidence Functions as Generalized Arithmetic Functions

4.1 Incidence Function

Theory of incidence functions provide a more general setting, compared to that of arithmetic functions, in which many of the properties of arithmetic functions, or more precisely, the generalizations and analogues of these properties hold in a more or less straightforward fashion. Within this framework incidence functions can be viewed as generalized arithmetic functions associated with a theory generalized from the theory of arithmetic functions.

The following introduction to incidence functions as generalized arithmetic functions is based on the main lines of the development of the associated theory that was initially presented by D. A. Smith [47], [48], and [49], and later restated and supplemented by P. J. McCarthy [25], where all the main results are presented in a context of a locally finite partially ordered set $(P, \leq)$ (see also [36, Chapter 2]). Since the poset $(P, \leq)$ need not be explicitly equipped with anything comparable to the arithmetic of integers, it entails that the focus is instead on the more general properties of lattices.

This introduction to incidence functions concentrates on the properties of factorability and complete factorability of incidence functions. Despite their very straightforward nature, the addition and multiplication of incidence functions are introduced briefly. Several convolutions of arithmetic functions, including the Dirichlet convolution and the unitary convolution, have been presented in the context of incidence functions (see e.g. [47, pp. 628–630]). However, this introduction covers only one of these convolutions, namely the generalization of the Dirichlet convolution.

Initially, in [47] D. A. Smith presented the concept of factorability of an incidence function and the related results under the setting of a locally finite lattice (see [47, §§ 4–5]). However, in [48, pp. 16–17] he remarked that this concept and the related results can be extended, straightforwardly, to apply also in a more general setting of a locally finite local lattice, which subsequently, in [48] and [49], acts as the setting with additional property of local distributivity, when appropriate. Later, in [52] D. A. Smith reorganized the presentations of [47], [48], and [49] into a more coherent presentation with an arithmetical emphasis. In this presentation he essentially restates some of the main results within a context of a partially ordered set (P, π), where $(P, \leq_P)$ is a subposet of the standard order lattice $(\mathbb{N}, \leq)$ and π is a proper suborder of the partial ordering $\leq_P$.

Definition 4.1. Let $(P, \leq)$ be a locally finite partially ordered set, and let $\langle K, +, \cdot \rangle$ be a field. A function $f : P \times P \to K$ is an *incidence function*, with values in a field K, of the poset $(P, \leq)$ if

$$\forall\, x, y \in P : x \nleq y \Rightarrow f(x, y) = 0.$$

The set of incidence functions, with values in a field $\langle K, +, \cdot \rangle$, of the locally finite partially ordered set $(P, \leq)$ is denoted by $\mathbb{I}[P, K, \leq]$.

Remark. The rational numbers, the real numbers, and the complex numbers, associated with the corresponding binary operations of addition and multiplication, i.e. the algebraic structures $\langle \mathbb{Q}, +, \cdot \rangle$, $\langle \mathbb{R}, +, \cdot \rangle$, and $\langle \mathbb{C}, +, \cdot \rangle$, respectively, are examples of a field of characteristic 0.

The following three incidence functions, namely the *zero function*, the *zeta function*, and the *delta function*, can be defined on every poset $(P, \leq)$, and they are the analogues of the corresponding arithmetic functions.

Definition 4.2. The *zero function* $\mathit{0} \in \mathbb{I}[P, K, \leq]$ is defined as follows:

$$\mathit{0} : P \times P \to K : \mathit{0}(x, y) = 0.$$

Definition 4.3. The *zeta function* $\zeta \in \mathbb{I}[P, K, \leq]$ is defined as follows:

$$\zeta : P \times P \to K : \zeta(x, y) = \begin{cases} 1 & \text{if } x \leq y, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 4.4. The *delta function* $\delta \in \mathbb{I}[P, K, \leq]$ is defined as follows:

$$\delta : P \times P \to K : \delta(x, y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 4.5. Let $f \in \mathbb{I}[P, K, \leq]$. The *summatory function* F of the function f is defined as follows:

$$F : P \times P \to K : F(x, y) = \sum_{x \leq z \leq y} f(x, z).$$

The summatory functions of incidence functions are naturally incidence functions themselves.

4.2 Addition and Multiplication

The addition and multiplication of incidence functions are defined by following the convention that is typical for functions, and therefore they share all the usual properties associated to these operations.

Definition 4.6. Let $f, g \in \mathbb{I}[P, K, \leq]$. A function $h \in \mathbb{I}[P, K, \leq]$ is the *sum* of the functions f and g if

$$\forall\, x, y \in P : h(x, y) = f(x, y) + g(x, y).$$

Definition 4.7. The binary operation $+$ in the set $\mathbb{I}[P, K, \leq]$, i.e. the *(pointwise) addition of incidence functions*, is defined as follows:

$$+ : \mathbb{I}[P, K, \leq] \times \mathbb{I}[P, K, \leq] \to \mathbb{I}[P, K, \leq] : +(f, g) = h,$$

where the function h is the sum of the functions f and g, denoted by $f + g$.

Definition 4.8. Let $f, g \in \mathbb{I}[P, K, \leq]$. A function $h \in \mathbb{I}[P, K, \leq]$ is the *product* of the functions f and g if

$$\forall\, x, y \in P : h(x, y) = f(x, y)g(x, y).$$

Definition 4.9. The binary operation $\cdot$ in the set $\mathbb{I}[P, K, \leq]$, i.e. the *(pointwise) multiplication of incidence functions*, is defined as follows:

$$\cdot : \mathbb{I}[P, K, \leq] \times \mathbb{I}[P, K, \leq] \to \mathbb{I}[P, K, \leq] : \cdot\,(f, g) = h,$$

where the function h is the product of the functions f and g, denoted by fg.

Theorem 4.1. *The algebraic structure $\langle \mathbb{I}[P, K, \leq], +, \cdot \rangle$ is a commutative ring with unity.*

The zero function $\mathit{0} \in \mathbb{I}[P, K, \leq]$ is the identity element with respect to the addition of incidence functions, and the zeta function $\zeta \in \mathbb{I}[P, K, \leq]$ is the identity element with respect to the multiplication of incidence functions.

Theorem 4.2. *A function $f \in \mathbb{I}[P, K, \leq]$ has a multiplicative inverse if and only if*

$$\forall\, x, y \in P : x \leq y \Rightarrow f(x, y) \neq 0.$$

This multiplicative inverse f^{-1} is defined as follows:

$$f^{-1} : P \times P \to K : f^{-1}(x, y) = \begin{cases} f(x, y)^{-1} & \textit{if } x \leq y, \\ 0 & \textit{otherwise.} \end{cases}$$

4.3 Convolution

The convolution of incidence functions is a binary operation defined on the set of incidence functions, and it builds upon the underlying partial ordering. It is, in a sense, a generalization of the Dirichlet convolution of arithmetic functions.

Definition 4.10. Let $f, g \in \mathbb{I}[P, K, \leq]$. A function $h \in \mathbb{I}[P, K, \leq]$ is the *convolution* of the functions f and g if

$$\forall\, x, y \in P : h(x, y) = \sum_{x \leq z \leq y} f(x, z)g(z, y).$$

Definition 4.11. The binary operation $*$ in the set $\mathbb{I}[P, K, \leq]$, i.e. the *convolution of incidence functions*, is defined as follows:

$$* : \mathbb{I}[P, K, \leq] \times \mathbb{I}[P, K, \leq] \to \mathbb{I}[P, K, \leq] : *\,(f, g) = h,$$

where the function h is the convolution of the functions f and g, denoted by $f * g$.

Theorem 4.3. *Let $f \in \mathbb{I}[P, K, \leq]$. Then $F \in \mathbb{I}[P, K, \leq]$ is the summatory function of the function f if and only if*

$$F = f * \zeta, \quad \text{where } \zeta \in \mathbb{I}[P, K, \leq].$$

Theorem 4.4. *The algebraic structure $\langle \mathbb{I}[P, K, \leq], +, * \rangle$ is a ring with unity.*

The zero function $0 \in \mathbb{I}[P, K, \leq]$ is the identity element with respect to the addition of incidence functions, and the delta function $\delta \in \mathbb{I}[P, K, \leq]$ is the identity element with respect to the convolution of incidence functions.

Theorem 4.5. *Let $(P, \leq)$ be a locally finite partially ordered set, and let $\langle K, +, \cdot \rangle$ be a field. Then the convolution of incidence functions defined on the set $\mathbb{I}[P, K, \leq]$ is commutative if and only if the poset $(P, \leq)$ is an antichain.*

Theorem 4.6. *A function $f \in \mathbb{I}[P, K, \leq]$ has a convolution inverse if and only if*

$$\forall\, x \in P : f(x, x) \neq 0.$$

This convolution inverse f^{*-1} is defined recursively as follows:

$$f^{*-1}(x, y) = \begin{cases} f(x, y)^{-1} & \text{if } x = y, \\ -\, f(y, y)^{-1} \displaystyle\sum_{x \leq z < y} f^{*-1}(x, z) f(z, y) & \text{if } x < y, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4.7. *If $f, g \in \mathbb{I}[P, K, \leq]$ are invertible with respect to the convolution, then $f * g$ is invertible with respect to the convolution.*

Theorem 4.8. *If $h \in \mathbb{I}[P, K, \leq]$ is invertible with respect to the convolution, then*

(i) $\forall\, f, g \in \mathbb{I}[P, K, \leq] : f = g * h \Leftrightarrow g = f * h^{*-1}$,

(ii) $\forall\, f, g \in \mathbb{I}[P, K, \leq] : f = h * g \Leftrightarrow g = h^{*-1} * f$.

Theorem 4.9. *Let $f \in \mathbb{I}[P, K, \leq]$. If $F \in \mathbb{I}[P, K, \leq]$ is the summatory function of the function f, then*

$$f = F * \zeta^{*-1}, \quad \textit{where } \zeta \in \mathbb{I}[P, K, \leq].$$

4.4 The Möbius Function

The Möbius function of the poset P, denoted by μ, is the analogue of the corresponding arithmetic function.

Definition 4.12. The *Möbius function* $\mu \in \mathbb{I}[P, K, \leq]$ is defined as follows:

$$\mu = \zeta^{*-1}, \quad \text{where } \zeta \in \mathbb{I}[P, K, \leq].$$

The following result, which can be taken as the defining property of the Möbius function, follows from the fact that $\mu * \zeta = \delta$.

Theorem 4.10. *The delta function $\delta \in \mathbb{I}[P, K, \leq]$ is the summatory function of the Möbius function $\mu \in \mathbb{I}[P, K, \leq]$, i.e.*

$$\sum_{x \leq z \leq y} \mu(x, z) = \begin{cases} 1 & \textit{if } x = y, \\ 0 & \textit{otherwise.} \end{cases}$$

Theorem 4.11. *The Möbius function $\mu \in \mathbb{I}[P, K, \leq]$ is determined recursively as follows:*

$$\mu(x, y) = \begin{cases} 1 & \textit{if } x = y, \\ -1 & \textit{if } x \prec y, \\ -\displaystyle\sum_{x \leq z < y} \mu(x, z) & \textit{otherwise.} \end{cases}$$

Theorem 4.12. *The Möbius inversion formulas for $\mathbb{I}[P, K, \leq]$ are the following:*

(i) $\forall\, f, g \in \mathbb{I}[P, K, \leq] : f = g * \zeta \Leftrightarrow g = f * \mu$,

(ii) $\forall\, f, g \in \mathbb{I}[P, K, \leq] : f = \zeta * g \Leftrightarrow g = \mu * f$.

The following theorem presents the Möbius inversion formulas, introduced in Theorem 4.12, in a different form.

Theorem 4.13. *The following hold for every* $f, g \in \mathbb{I}[P, K, \leq]$*:*

(i) $\forall\, x, y \in P : f(x, y) = \sum_{x \leq z \leq y} g(x, z)$
if and only if
$\forall\, x, y \in P : g(x, y) = \sum_{x \leq z \leq y} f(x, z)\mu(z, y)$,

(ii) $\forall\, x, y \in P : f(x, y) = \sum_{x \leq z \leq y} g(z, y)$
if and only if
$\forall\, x, y \in P : g(x, y) = \sum_{x \leq z \leq y} \mu(x, z) f(z, y)$.

4.5 The Cardinality Function

The cardinality function of the poset P, denoted by $\tau \in \mathbb{I}[P, K, \leq]$, is the analogue of the corresponding arithmetic function.

Definition 4.13. The *cardinality function* $\tau \in \mathbb{I}[P, K, \leq]$ is defined as follows:

$$\tau = \zeta * \zeta, \quad \text{where } \zeta \in \mathbb{I}[P, K, \leq].$$

For all $x, y \in P$,

$$\tau(x, y) = \sum_{x \leq z \leq y} \zeta(x, z)\zeta(z, y) = \sum_{x \leq z \leq y} 1,$$

i.e. the value of $\tau(x, y)$ is determined by the number of elements in the interval $[\, x, y \,]$, which motivates to call this function the cardinality function.

Theorem 4.14. *The cardinality function* $\tau \in \mathbb{I}[P, K, \leq]$ *is the summatory function of the zeta function* $\zeta \in \mathbb{I}[P, K, \leq]$*.*

Remark. In [47] D. A. Smith presented also the power functions, the divisor functions, and the Euler function in the context of incidence functions (see [47, pp. 626–628]).

4.6 Factorability

The notion of factorability of an incidence function of a locally finite local lattice $(P, \leq)$ generalizes the notion of multiplicativity of an arithmetic function. This generalization builds upon the underlying partial ordering $\leq$ and the related lattice operations meet $(\wedge)$ and join $(\vee)$.

Definition 4.14. Let $(P, \leq)$ be a locally finite local lattice, and let $\langle K, +, \cdot \rangle$ be a field. A function $f \in \mathbb{I}[P, K, \leq]$ is *factorable* if

(i) $\forall\, x \in P : f(x, x) = 1$,

(ii) $\forall\, x, y, z, w \in P :$
$$\Big[\big[\exists\, u, v \in P : x, y, z, w \in [\, u, v \,]\big],\ x \leq y,\ z \leq w,\ \text{and } x \wedge z = y \wedge w\Big]$$
$$\Rightarrow f(x \vee z, y \vee w) = f(x, y) f(z, w).$$

Remark. M. Ward [57] generalized the result of Theorem 3.13 to incidence functions, and called an incidence function of a lattice $(L, \leq)$ factorable if

$$\begin{aligned} &\forall\, x, y, z, w \in P : \Big[x \leq y \text{ and } z \leq w\Big] \\ &\qquad \Rightarrow f(x \wedge z, y \wedge w) f(x \vee z, y \vee w) = f(x, y) f(z, w). \end{aligned}$$

Almost three decades later, D. A. Smith [47], [48], [49], and [52] weakened the definition of factorability into its present form, formulated by Definition 4.14, in order to arrive at the results that follow. Initially, in place of the condition (i) of Definition 4.14 D. A. Smith required that a factorable function is invertible with respect to the convolution, which actually together with the condition (ii) implies the condition (i) (see [47, p. 622], and also [17]).

The zeta function $\zeta \in \mathbb{I}[P, K, \leq]$ is factorable regardless of the structure of the related locally finite local lattice.

Theorem 4.15. *Let $(P, \leq)$ be a locally finite local lattice. Then the delta function $\delta \in \mathbb{I}[P, K, \leq]$ is factorable if and only if $(P, \leq)$ is locally modular.*

Theorem 4.16. *If $(P, \leq)$ is a locally finite locally distributive local lattice, then*

$$\forall\, f, g \in \mathbb{I}[P, K, \leq] : f \textit{ and } g \textit{ are factorable } \Rightarrow f * g \textit{ is factorable.}$$

Theorem 4.17. *If $(P, \leq)$ is a locally finite locally distributive local lattice, then*

$$\forall\, f \in \mathbb{I}[P, K, \leq] : f \textit{ is factorable } \Rightarrow f^{*-1} \textit{ is factorable.}$$

The converses of Theorems 4.16 and 4.17 do not hold in general for an arbitrary locally finite local lattice and an arbitrary field, as is reflected by the following theorems.

Theorem 4.18. *Let $(P, \leq)$ be a locally finite local lattice, and let $\langle K, +, \cdot\rangle$ be a field of characteristic 0. If the cardinality function $\tau \in \mathbb{I}[P, K, \leq]$ is factorable, then $(P, \leq)$ is locally distributive.*

Theorem 4.19. *Let $(P, \leq)$ be a locally finite local lattice, and let $\langle K, +, \cdot\rangle$ be a field of characteristic 0. If the Möbius function $\mu \in \mathbb{I}[P, K, \leq]$ is factorable, then $(P, \leq)$ is locally distributive.*

Theorem 4.20. *Let $(P, \leq)$ be a locally finite local lattice, and let $\langle K, +, \cdot\rangle$ be a field of characteristic 0. Then*

$$\forall\, f, g \in \mathbb{I}[P, K, \leq] : f \textit{ and } g \textit{ are factorable } \Rightarrow f * g \textit{ is factorable}$$

if and only if $(P, \leq)$ is locally distributive.

Theorem 4.21. *Let* $(P, \leq)$ *be a locally finite local lattice, and let* $\langle K, +, \cdot \rangle$ *be a field of characteristic* 0*. Then*

$$\forall f \in \mathbb{I}[P, K, \leq] : f \text{ is factorable } \Rightarrow f^{*-1} \text{ is factorable}$$

if and only if $(P, \leq)$ *is locally distributive.*

Theorem 4.22. *Let* $(P, \leq)$ *be a locally finite locally distributive local lattice, and let* $\langle K, +, \cdot \rangle$ *be a field of characteristic* 0*. If* $F \in \mathbb{I}[P, K, \leq]$ *is the summatory function of the function* $f \in \mathbb{I}[P, K, \leq]$*, then* f *is factorable if and only if* F *is factorable.*

Remark. Theorem 4.22 is, in fact, a generalization of a result that D. A. Smith incidentally pointed out (see [47, p. 624]).

As the previous theorems indicate, the notion of factorability rests heavily on the local distributivity of the underlying local lattice.

Theorem 4.23. *Let* $(P, \leq)$ *be a locally finite locally distributive local lattice. A function* $f \in \mathbb{I}[P, K, \leq]$ *is factorable if and only if*

(i) $\forall x \in P : f(x, x) = 1,$

(ii) $$\forall x, y, w \in P : \Big[\big[\exists u, v \in P : x, y, w \in [\,u, v\,] \big] \text{ and } x = y \wedge w \Big] \Rightarrow f(x, y \vee w)) = f(x, y) f(x, w).$$

(iii) $$\forall x, y, z \in P : \Big[\big[\exists u, v \in P : x, y, z \in [\,u, v\,] \big],\ x \leq y, \text{ and } x \wedge z = y \wedge z \Big] \Rightarrow f(x \vee z, y \vee z) = f(x, y).$$

4.7 Translation Invariance

The conditions (i)–(iii) presented by Theorem 4.23 suggest two properties that can be used in classifying the incidence functions. First, as a special case of factorability with $x = z$, the conditions (i) and (ii) together are a reminiscent of the notion of factorability, and therefore suggest a weaker property which could be labeled as the 'semifactorability'. Second, the condition (iii), which together with the condition (i) is a special case of factorability with $z = w$, instead, lacks the essential features of the factorability, and therefore suggests a different kind of notion that is labeled as the 'translation invariance'.

Remark. D. A. Smith [47]–[52] and P. J. McCarthy [25] do not use the term 'semifactorability'.

Definition 4.15. Let $(P, \leq)$ be a locally finite local lattice. A function $f \in \mathbb{I}[P, K, \leq]$ is *translation invariant* if

$$\forall\, x, y, z \in P : \Big[\big[\exists\, u, v \in P : x, y, z \in [\, u, v\,]\big],\ x \leq y,\ \text{and } x \wedge z = y \wedge z\Big]$$
$$\Rightarrow f(x \vee z, y \vee z) = f(x, y).$$

The zeta function $\zeta \in \mathbb{I}[P, K, \leq]$ is translation invariant regardless of the structure of the related locally finite local lattice.

Theorem 4.24. *Let $(P, \leq)$ be a locally finite locally distributive local lattice. A function $f \in \mathbb{I}[P, K, \leq]$ is translation invariant if and only if*

$$\forall\, x, y \in P : \big[\exists\, u, v \in P : x, y \in [\, u, v\,]\big] \Rightarrow f(x \wedge y, x) = f(y, x \vee y).$$

As the previous theorem suggests, also the notion of translation invariance of an incidence function is closely connected to and depends heavily on the distributivity of the underlying local lattice.

4.8 Complete Factorability

The notion of complete factorability of an incidence function of a locally finite local lattice $(P, \leq)$ generalizes the notion of complete multiplicativity of an arithmetic function, and it builds upon the concept of factorability by strengthening the requirements.

Definition 4.16. Let $(P, \leq)$ be a locally finite local lattice. A function $f \in \mathbb{I}[P, K, \leq]$ is *completely factorable* if it is translation invariant and

(i) $\forall\, x \in P : f(x, x) = 1$,

(ii) $\forall\, x, y, z \in P : x \leq z \leq y \Rightarrow f(x, y) = f(x, z) f(z, y)$.

Remark. Definition 4.16 presents the concept of complete factorability, in essence, as it was introduced by D. A. Smith [49, p. 357]. Later, however, Smith [52, p. 212] applied a weakened notion of complete factorability in a more specific context of a partially ordered set (P, π), where π is a proper suborder of the partial ordering $\leq_P$ and $(P, \leq_P)$ is a subposet of the standard order lattice $(\mathbb{N}, \leq)$, and called an incidence function completely factorable if

$$\forall\, x, y, z \in P : x \;\pi\; z \;\pi\; y \Rightarrow f(x, y) = f(x, z) f(z, y).$$

Subsequently, P. J. McCarthy [25, p. 320] applied this weaker definition of complete factorability in a more general context of locally finite poset $(P, \leq)$.

Theorem 4.25. *Let $(P, \leq)$ be a locally finite locally distributive local lattice, and let $f \in \mathbb{I}[P, K, \leq]$. If the function f is translation invariant and*

$$\forall\, x, y, z \in P : x \leq z \leq y \Rightarrow f(x, y) = f(x, z)f(z, y),$$

then

$$\forall\, x, y, w \in P : \Big[\big[\exists\, u, v \in P : x, y, w \in [\,u, v\,]\big] \text{ and } x = y \wedge w\Big] \\ \Rightarrow f(x, y \vee w) = f(x, y)f(x, w).$$

Theorem 4.26. *Let $(P, \leq)$ be a locally finite locally distributive local lattice. If $f \in \mathbb{I}[P, K, \leq]$ is completely factorable, then it is factorable.*

Remark. If the weaker definition of complete factorability (see previous remark) is applied, then the translation invariance of a function $f \in \mathbb{I}[P, K, \leq]$ should be included in the hypothesis of Theorem 4.26. In fact, this is exactly how D. A. Smith [52, p. 237] and P. J. McCarthy [25, p. 322] deal with the case. Nevertheless, this kind of escape route does not resolve the inherent problem with the weaker definition of complete factorability, namely that the complete factorability of a function, in itself, is not sufficient to establish the factorability of that function.

Theorem 4.27. *Let $f \in \mathbb{I}[P, K, \leq]$. Then*

$$\forall\, x, y, z \in P : x \leq z \leq y \Rightarrow f(x, y) = f(x, z)f(z, y)$$

if and only if

$$\forall\, g, h \in \mathbb{I}[P, K, \leq] : f(g * h) = (fg) * (fh).$$

Remark. Theorem 4.27 was first introduced by P. J. McCarthy [24].

Theorem 4.28. *If $f \in \mathbb{I}[P, K, \leq]$ is such that*

(i) $\forall\, x \in P : f(x, x) = 1$,

(ii) $\forall\, x, y, z \in P : x \leq z \leq y \Rightarrow f(x, y) = f(x, z)f(z, y)$.

then $f^{*-1} = f\mu$.

Chapter 5

Arithmetic Incidence Functions

5.1 Arithmetic Incidence Function

To begin with, the term *arithmetic incidence function* is introduced to imply that the subject matter is the standard order lattice $(\mathbb{Z}_+, \leq)$ and, more precisely, its incidence functions. First of all, as the term itself suggests, an arithmetic incidence function is a specific instantiation of the general concept of an incidence function, and therefore can also be treated as such. On the other hand, the concept of an arithmetic incidence function is, essentially, also a natural generalization of the concept of an arithmetic function.

Definition 5.1. A function $f : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C}$ is an *arithmetic incidence function* if

$$\forall\, x, y \in \mathbb{Z}_+ : x \nleq y \Rightarrow f(x, y) = 0.$$

Following the adopted notation, the set of arithmetic incidence functions is denoted by $\mathbb{I}[\mathbb{Z}_+, \mathbb{C}, \leq]$, or more simply by $\mathbb{I}[\mathbb{Z}_+, \leq]$ if no confusion about the codomain can arise, which is the case in this presentation.

Remark. As in the case of incidence functions an arbitrary field could act as the codomain of an arithmetic incidence function, but the decision here is to use the field of complex numbers.

The seminal works of D. A. Smith [47], [48] [49], and [52], restated by P. J. McCarthy [25, Chapter 7], show that the incidence functions can be treated, from a certain viewpoint, as generalized arithmetic functions. These works also deal with the subject of arithmetic incidence functions, as described above, but the focus is primarily on the general setting of incidence functions. This more general setting, as the framework and the starting point of the generalization, inevitably lacks some of the essential features of the more specific setting provided by the arithmetic of integers, and this fact is also reflected in the resulting generalized theory.

Remark. D. A. Smith [47], [48], [49], and [52] and P. J. McCarthy [25] do not use the term 'arithmetic incidence function'.

Since the domain of an arithmetic incidence function is the Cartesian product of integers, it follows that, regarding the function arguments, the arithmetic of integers is fully applicable for various arithmetical speculations and manipulations, and especially in establishing different arithmetical relationships between the function arguments. Holding tight with this asset, the arithmetic incidence functions set up a more specific setting, compared to that of incidence functions, for the generalization of different features of the theory of arithmetic functions.

The arithmetic incidence functions form a subclass of the class of arithmetic functions of two variables. Correspondingly, the various subsets of the set $\mathbb{I}[\mathbb{Z}_+, \leq]$ can be regarded as subclasses of the class of arithmetic incidence functions. Of special interest are the subclasses which are determined by the suborders of the standard ordering of positive integers (see Definition 2.14), and specifically such subclasses where the underlying partial ordering constitutes a structure of a lattice. In the center of interest is the subclass of arithmetic incidence functions which is determined by the divisibility ordering of the set of positive integers. In effect, this subclass is formed by the incidence functions of the factor lattice $(\mathbb{Z}_+, \trianglelefteq)$, and therefore, following the adopted convention, it is denoted by $\mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$.

The defining property of arithmetic incidence functions which separates them from the rest of the arithmetic functions of two variables is utilized in the following theorem presenting a necessary and a sufficient condition for the equalness of two arithmetic incidence functions.

Lemma 5.1. *Let $f, g \in \mathbb{I}[\mathbb{Z}_+, \leq]$. Then $f = g$ if and only if*

$$\forall\, x, y \in \mathbb{Z}_+ : x \leq y \Rightarrow f(x, y) = g(x, y).$$

Proof. Elementary. □

In order to decide the equalness of two incidence functions belonging to a same subclass, in light of Lemma 5.1, it is sufficient to investigate the function values only at comparable elements. In the forthcoming, this strategy is employed when seen as convenient and appropriate, and moreover, due to its very basic nature, without any explicit mention.

As any class of functions, the class of arithmetic incidence functions is equipped with various binary operations. In addition to the pointwise addition and the pointwise multiplication, also many types of convolutions of arithmetic functions can be generalized as binary operations of arithmetic incidence functions. As an example of such binary operation is the *S-convolution* which is a generalization of the *Cauchy convolution* of arithmetic functions (exceptionally defined in the set $\mathbb{N}$, see e.g. [55]).

Definition 5.2. Let $f, g \in \mathbb{I}[\mathbb{Z}_+, \leq]$. A function $h \in \mathbb{I}[\mathbb{Z}_+, \leq]$ is the *S-convolution* of the functions f and g if

$$\forall\, x, y \in \mathbb{Z}_+ : h(x,y) = \sum_{\substack{x \leq z \leq y \\ z+w=x+y}} f(x,z)g(x,w),$$

where the sum runs over all ordered pairs $\langle z, w\rangle$ of positive integers satisfying $x \leq z \leq y$ and $z + w = x + y$.

Remark. The letter 'S' in S-convolution refers to 'standard ordering'.

Definition 5.3. The binary operation $*_S$ in the set $\mathbb{I}[\mathbb{Z}_+, \leq]$, i.e. the *S-convolution of arithmetic incidence functions*, is defined as follows:

$$*_S : \mathbb{I}[\mathbb{Z}_+, \leq] \times \mathbb{I}[\mathbb{Z}_+, \leq] \to \mathbb{I}[\mathbb{Z}_+, \leq] : *_S\,(f, g) = h,$$

where the function h is the S-convolution of the functions f and g, denoted by $f *_S g$.

Binary operations can be defined either in the class of arithmetic incidence functions or locally in its subclasses. If a binary operation is defined in the class of arithmetic incidence functions, then it is not necessary a binary operation in all its subclasses. On the other hand, if a binary operation is defined in some proper subclass, then it is also extensible to the class of arithmetic incidence functions while preserving all its properties. The pointwise addition and pointwise multiplication defined in the class of arithmetic incidence functions are naturally binary operations also in all its subclasses.

Theorem 5.1. *The S-convolution of arithmetic incidence functions is a binary operation also in the subclass* $\mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$.

Proof. Let $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. Let us assume that $x, y \in \mathbb{Z}_+$ are such that $(f *_S g)(x, y) \neq 0$. Then there are $z, w \in \mathbb{Z}_+$ for which it holds that $x \leq z \leq y$, $z + w = x + y$, $f(x, z) \neq 0$, and $g(x, w) \neq 0$. Since $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, it follows that $x \trianglelefteq z$ and $x \trianglelefteq w$, and therefore $x \mid z + w$. Thus $x \mid x + y$, and $x \mid -x$, and therefore $x \trianglelefteq y$. This establishes that if $(f *_S g)(x, y) \neq 0$, then $x \trianglelefteq y$. By contraposition principle it follows that if $x \ntrianglelefteq y$, then $(f *_S g)(x, y) = 0$, which establishes that $\mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is closed under the S-convolution. $\square$

The concept of an arithmetic incidence function can be generalized also to functions of more than two variables.

Definition 5.4. A function $f : \mathbb{Z}_+^n \to \mathbb{C}$, where $n \geq 2$, is an *arithmetic incidence function of* n *variables* if

$$\forall\, x_1, x_2, \ldots, x_n \in \mathbb{Z}_+ : \neg\,(x_1 \leq x_2 \leq \cdots \leq x_n) \Rightarrow f(x_1, x_2, \ldots, x_n) = 0.$$

The present introduction to arithmetic incidence functions deals with the arithmetic incidence functions of two variables, simply referred to as the arithmetic incidence functions, and a selection of associated properties that originate from the underlying divisibility properties of integers. Specifically, in the center of interest are the generalizations of the concepts of multiplicativity and complete multiplicativity of arithmetic functions of one variable to arithmetic incidence functions.

The focus being on properties originating from the divisibility of integers, the primary investigations can be directed to the subclass $\mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, since it is the smallest subclass of arithmetic incidence functions that covers all possible value combinations regarding the function arguments which are in divisibility relation. However, with some notable exceptions, many of the divisibility based results that hold in the subclass $\mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, hold also in the main class of arithmetic incidence functions, and even in the class of arithmetic functions of two variables.

It is also evident that certain parts of the theory of arithmetic functions, to a large extent, can be embedded in the theory of arithmetic incidence functions without any major adjustments. In other words, the arithmetic incidence functions offer a method to generalize or extend the theory of arithmetic functions.

In the following, in light of the above observations, all definitions and associated results are given in the context of the subclass $\mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, while acknowledging the possibility to extend them to apply also in the class of arithmetic incidence functions. Moreover, taking this restricted context into account, the term 'arithmetic incidence function' is used to refer the arithmetic incidence functions of the subclass $\mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$.

5.2 Some Examples of Arithmetic Incidence Functions

The following arithmetic incidence functions are generalizations of the corresponding arithmetic functions.

Definition 5.5. The function $\gamma \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is defined as follows:

$$\gamma : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C} : \gamma(x, y) = \begin{cases} \prod\limits_{\substack{px \trianglelefteq y \\ p \in \mathbb{P}}} p & \text{if } x \trianglelefteq y, \\ 0 & \text{otherwise,} \end{cases}$$

i.e. the value of $\gamma(x, y)$ is the product of distinct prime factors of y/x.

Definition 5.6. The function $\omega \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is defined as follows:

$$\omega : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C} : \omega(x, y) = \begin{cases} \displaystyle\sum_{\substack{px \trianglelefteq y \\ p \in \mathbb{P}}} 1 & \text{if } x \trianglelefteq y, \\ 0 & \text{otherwise,} \end{cases}$$

i.e. the value of $\omega(x, y)$ is the number of distinct prime factors of y/x.

Definition 5.7. The function $\Omega \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is defined as follows:

$$\Omega : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C} : \Omega(x, y) = \begin{cases} \displaystyle\sum_{\substack{px \trianglelefteq y \\ p \in \mathbb{P}}} [y(p) - x(p)] & \text{if } x \trianglelefteq y, \\ 0 & \text{otherwise,} \end{cases}$$

where $x = \prod_{p \in \mathbb{P}} p^{x(p)}$ and $y = \prod_{p \in \mathbb{P}} p^{y(p)}$, i.e. the value of $\Omega(x, y)$ is the total number of prime factors of y/x, each counted according to its multiplicity.

The *zero function*, the *zeta function*, and the *delta function* of the factor lattice $(\mathbb{Z}_+, \trianglelefteq)$ are generalizations of the corresponding arithmetic functions.

Definition 5.8. The *zero function* $\mathit{0} \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is defined as follows:

$$\mathit{0} : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C} : \mathit{0}(x, y) = 0.$$

Definition 5.9. The *zeta function* $\zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is defined as follows:

$$\zeta : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C} : \zeta(x, y) = \begin{cases} 1 & \text{if } x \trianglelefteq y, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 5.10. The *delta function* $\delta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is defined as follows:

$$\delta : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C} : \delta(x, y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

Remark. If a subclass of arithmetic incidence functions is determined by some suborder of the standard ordering of positive integers (see Definition 2.14), then $0 \in \mathbb{I}[\mathbb{Z}_+, \leq]$ and $\delta \in \mathbb{I}[\mathbb{Z}_+, \leq]$ are the zero and the delta functions, respectively, also in this subclass. In contrast to this, the same does not apply to the zeta function which varies from subclass to subclass due to the underlying partial ordering.

Definition 5.11. A function $F \in \mathbb{I}[\mathbb{Z}_+, \leq]$ is the *(standard order) summatory function* of a function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ if

$$\forall\, x, y \in \mathbb{Z}_+ : F(x, y) = \sum_{x \leq z \leq y} f(x, z).$$

5.3 Addition

The addition of arithmetic incidence functions is defined by following the convention that is typical for functions. Despite the very straightforward nature of the addition of functions, a short presentation of the addition of arithmetic incidence functions is in place.

Definition 5.12. Let $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. A function $h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is the *sum* of the functions f and g if

$$\forall\, x, y \in \mathbb{Z}_+ : h(x, y) = f(x, y) + g(x, y).$$

Definition 5.13. The binary operation $+$ in the set $\mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, i.e. the *(pointwise) addition of arithmetic incidence functions*, is defined as follows:

$$+ : \mathbb{I}[\mathbb{Z}_+, \trianglelefteq] \times \mathbb{I}[\mathbb{Z}_+, \trianglelefteq] \to \mathbb{I}[\mathbb{Z}_+, \trianglelefteq] : +\,(f, g) = h,$$

where the function h is the sum of the functions f and g, denoted by $f + g$.

The addition of arithmetic incidence functions shares all the usual properties associated to the addition of functions.

Theorem 5.2. *The addition of arithmetic incidence functions is associative.*

Proof. Let $f, g, h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ and $x, y \in \mathbb{Z}_+$. Then

$$\begin{aligned}[(f+g)+h](x, y) &= (f+g)(x, y) + h(x, y) = (f(x, y) + g(x, y)) + h(x, y) \\ &= f(x, y) + (g(x, y) + h(x, y)) = f(x, y) + (g+h)(x, y) \\ &= [f + (g+h)](x, y).\end{aligned}$$

Thus $(f+g)+h = f+(g+h)$, and therefore the addition of arithmetic incidence functions is associative. □

Theorem 5.3. *The addition of arithmetic incidence functions is commutative.*

Proof. Let $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ and $x, y \in \mathbb{Z}_+$. Then

$$(f+g)(x, y) = f(x, y) + g(x, y) = g(x, y) + f(x, y) = (g+f)(x, y).$$

Thus $f + g = g + f$, and therefore the addition of arithmetic incidence functions is commutative. □

Theorem 5.4. *The zero function $0 \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is the identity element with respect to the addition of arithmetic incidence functions.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ and $x, y \in \mathbb{Z}_+$. Then

$$(f + \mathit{0})(x, y) = f(x, y) + \mathit{0}(x, y) = f(x, y) + 0 = f(x, y).$$

Thus $f + \mathit{0} = f$. Since $f + \mathit{0} = f$, it follows by Theorem 5.3 that $\mathit{0} + f = f$. Thus $\mathit{0} \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is the identity element with respect to the addition of arithmetic incidence functions. □

The additive inverse of an arithmetic incidence function f is denoted by $-f$.

Theorem 5.5. *If $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, then it has an additive inverse. This additive inverse $-f$ is defined as follows:*

$$-f : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C} : (-f)(x, y) = -f(x, y).$$

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, and let us define the function $g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ as follows:

$$g : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C} : g(x, y) = -f(x, y).$$

Let $x, y \in \mathbb{Z}_+$. Then

$$(f + g)(x, y) = f(x, y) + g(x, y) = f(x, y) + (-f(x, y)) = 0 = \mathit{0}(x, y).$$

Thus $f + g = \mathit{0}$. Since $f + g = \mathit{0}$, it follows by Theorem 5.3 that $g + f = \mathit{0}$. Thus g is the additive inverse of f, i.e. $g = -f$. □

Theorem 5.6. *The algebraic structure $\langle \mathbb{I}[\mathbb{Z}_+, \trianglelefteq], + \rangle$ is a commutative group.*

Proof. Follows by Theorems 5.2, 5.3, 5.4, and 5.5. □

Remark. Among the properties of arithmetic incidence functions that are to be introduced in the following only the translation invariance and the complete translation invariance are preserved by the addition of arithmetic incidence functions and shared by two arithmetic incidence functions which are additive inverses of each other.

5.4 Multiplication

The multiplication of arithmetic incidence functions is defined by following the convention that is typical for functions. Despite the very straightforward nature of the multiplication of functions, a short presentation of the multiplication of arithmetic incidence functions is in place.

Definition 5.14. Let $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. A function $h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is the *product* of the functions f and g if

$$\forall\, x, y \in \mathbb{Z}_+ : h(x, y) = f(x, y)g(x, y).$$

Definition 5.15. The binary operation $\cdot$ in the set $\mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, i.e. the *(pointwise) multiplication of arithmetic incidence functions*, is defined as follows:

$$\cdot : \mathbb{I}[\mathbb{Z}_+, \trianglelefteq] \times \mathbb{I}[\mathbb{Z}_+, \trianglelefteq] \to \mathbb{I}[\mathbb{Z}_+, \trianglelefteq] : \cdot (f, g) = h,$$

where the function h is the product of the functions f and g, denote by fg.

Following the usual convention, the multiplication of arithmetic incidence functions is denoted by juxtaposition, i.e. the notation fg denotes the multiplication of f and g. However, if the clarity of presentation so requires, the notation $\cdot$ is used instead of juxtaposition.

The multiplication of arithmetic incidence functions shares all the usual properties associated to the multiplication of functions.

Theorem 5.7. *The multiplication of arithmetic incidence functions is associative.*

Proof. Let $f, g, h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ and $x, y \in \mathbb{Z}_+$. Then

$$\begin{aligned}
[(fg)h](x, y) &= (fg)(x, y)h(x, y) \\
&= (f(x, y)g(x, y))h(x, y) \\
&= f(x, y)(g(x, y)h(x, y)) \\
&= f(x, y)(gh)(x, y) \\
&= [f(gh)](x, y).
\end{aligned}$$

Thus $(fg)h = f(gh)$, and therefore the multiplication of arithmetic incidence functions is associative. □

Theorem 5.8. *The multiplication of arithmetic incidence functions is commutative.*

Proof. Let $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ and $x, y \in \mathbb{Z}_+$. Then

$$(fg)(x, y) = f(x, y)g(x, y) = g(x, y)f(x, y) = (gf)(x, y).$$

Thus $fg = gf$, and therefore the multiplication of arithmetic incidence functions is commutative. □

Theorem 5.9. *The zeta function $\zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is the identity element with respect to the multiplication of arithmetic incidence functions.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ and $x, y \in \mathbb{Z}_+$. Then

$$(f\zeta)(x, y) = f(x, y)\zeta(x, y) = f(x, y) \cdot 1 = f(x, y).$$

Thus $f\zeta = f$. Since $f\zeta = f$, it follows by Theorem 5.8 that $\zeta f = f$. Thus $\zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is the identity element with respect to the multiplication of arithmetic incidence functions. □

Theorem 5.10. *The algebraic structure $\langle \mathbb{I}[\mathbb{Z}_+, \trianglelefteq], \cdot \rangle$ is a commutative semigroup with identity element.*

Proof. Follows by Theorems 5.7, 5.8, and 5.9. $\square$

If $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is invertible with respect to the multiplication, then its multiplicative inverse is denoted by f^{-1}. Since the zeta function $\zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is the multiplicative identity element, it is invertible with respect to the multiplication being its own inverse, i.e. $\zeta^{-1} = \zeta$. Let us next introduce a necessary and sufficient condition for an arithmetic incidence function to have a multiplicative inverse.

Theorem 5.11. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ has a multiplicative inverse if and only if*

$$\forall\, x, y \in \mathbb{Z}_+ : x \trianglelefteq y \Rightarrow f(x,y) \neq 0.$$

This multiplicative inverse f^{-1} is defined as follows:

$$f^{-1} : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C} : f^{-1}(x,y) = \begin{cases} f(x,y)^{-1} & \textit{if } x \trianglelefteq y, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. Let us first assume that f has a multiplicative inverse. Then $f \cdot f^{-1} = \zeta$, where $f^{-1} \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is the inverse of f. Let $x, y \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$. Let us assume that $f(x,y) = 0$. Then

$$\zeta(x,y) = (f \cdot f^{-1})(x,y) = f(x,y) f^{-1}(x,y) = 0 \cdot f^{-1}(x,y) = 0.$$

This contradicts the fact that $\zeta(x,y) = 1$, and therefore $f(x,y) \neq 0$. Thus

$$\forall\, x, y \in \mathbb{Z}_+ : x \trianglelefteq y \Rightarrow f(x,y) \neq 0.$$

Let us next assume that

$$\forall\, x, y \in \mathbb{Z}_+ : x \trianglelefteq y \Rightarrow f(x,y) \neq 0$$

and define the function $g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ as follows:

$$g : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C} : g(x,y) = \begin{cases} f(x,y)^{-1}, & \text{if } x \trianglelefteq y; \\ 0, & \text{otherwise.} \end{cases}$$

Let $x, y \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$. Then

$$(fg)(x,y) = f(x,y) g(x,y) = f(x,y) f(x,y)^{-1} = 1 = \zeta(x,y).$$

Thus $fg = \zeta$. Since $fg = \zeta$, it follows by Theorem 5.8 that $gf = \zeta$. Thus g is the multiplicative inverse of f, i.e. $g = f^{-1}$. $\square$

Theorem 5.12. *If $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ are invertible with respect to the multiplication, then fg is invertible with respect to the multiplication.*

Proof. Let $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be invertible with respect to the multiplication, and let $x, y \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$. Then by Theorem 5.11 $f(x, y) \neq 0$ and $g(x, y) \neq 0$, and therefore $(fg)(x, y) \neq 0$. Thus by Theorem 5.11 fg is invertible with respect to the multiplication. $\square$

Theorem 5.13. *The multiplication of arithmetic incidence functions distributes over the addition of arithmetic incidence functions.*

Proof. Let $f, g, h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ and $x, y \in \mathbb{Z}_+$. Then

$$\begin{aligned}
[f(g+h)](x, y) &= f(x, y)(g+h)(x, y) \\
&= f(x, y)(g(x, y) + h(x, y)) \\
&= f(x, y)g(x, y) + f(x, y)h(x, y) \\
&= (fg)(x, y) + (fh)(x, y) \\
&= [(fg) + (fh)](x, y).
\end{aligned}$$

Thus $f(g + h) = (fg) + (fh)$, and therefore the left distributive law holds. By Theorem 5.8 and by the left distributive law

$$(f + g)h = h(f + g) = (hf) + (hg) = (fh) + (gh),$$

and therefore the right distributive law holds. Thus the multiplication of arithmetic incidence functions distributes over the addition of arithmetic incidence functions. $\square$

As is usual for the addition and multiplication, the addition of arithmetic incidence functions does not distribute over the multiplication of arithmetic incidence functions. Consequently, following the usual convention, multiplication is performed before addition in the absence of parentheses, and therefore the related distributive laws take the following form:

$$f(g + h) = fg + fh \quad \text{and} \quad (f + g)h = fh + gh.$$

Theorem 5.14. *The algebraic structure $\langle \mathbb{I}[\mathbb{Z}_+, \trianglelefteq], +, \cdot \rangle$ is a commutative ring with unity.*

Proof. Follows by Theorems 5.6, 5.10, and 5.13. $\square$

Remark. The properties of arithmetic incidence functions that are to be introduced in the following are preserved by the multiplication of arithmetic incidence functions due to its straightforward definition. For the same reason, two arithmetic incidence functions which are multiplicative inverses of each other, possess a same subset of the aforementioned properties.

5.5 *D*-convolution

Despite the differences in their apperarances and essences, the convolution of incidence functions can be viewed as a generalization of the Dirichlet convolution of arithmetic functions. The *D-convolution* of arithmetic incidence functions, while preserving the essence of convolution of incidence functions by building upon the underlying partial ordering determined by the factor relation $\trianglelefteq$, rests profoundly also on the more specific divisibility relationships of integers. Consequently, the D-convolution offers another generalization of the Dirichlet convolution.

Lemma 5.2. *Let $x, y, z \in \mathbb{Z}_+$ be such that $x \trianglelefteq z \trianglelefteq y$. Then*

$$\exists\, w \in \mathbb{Z}_+ : x \trianglelefteq w \trianglelefteq y \text{ and } zw = xy.$$

Proof. Let $x, y, z \in \mathbb{Z}_+$ be such that $x \trianglelefteq z \trianglelefteq y$. Then $z = xu$ and $y = zv$, where $u, v \in \mathbb{Z}_+$. Let $w \in \mathbb{Z}_+$ be such that $w = xv$. Then

$$y = zv = xuv = xvu = wu \quad \text{and} \quad zw = zxv = xzv = xy.$$

Thus $x \trianglelefteq w \trianglelefteq y$ and $zw = xy$. $\square$

Lemma 5.3. *Let $x, y, z, w \in \mathbb{Z}_+$ be such that $zw = xy$. Then*

(i) *$x \trianglelefteq z \trianglelefteq y$ if and only if $x \trianglelefteq w \trianglelefteq y$,*

(ii) *$x \triangleleft z \trianglelefteq y$ if and only if $x \trianglelefteq w \triangleleft y$.*

Proof. (i) Let $x, y, z, w \in \mathbb{Z}_+$ be such that $zw = xy$. Let us first assume that $x \trianglelefteq z \trianglelefteq y$. Then $z = xu$ and $y = zv$, where $u, v \in \mathbb{Z}_+$. From $zw = xy$ and $y = zv$ it follows that $w = xv$, and likewise from $zw = xy$ and $z = xu$ it follows that $y = wu$. Thus $x \trianglelefteq w \trianglelefteq y$. Let us next assume that $x \trianglelefteq w \trianglelefteq y$. Then $w = xu$ and $y = wv$, where $u, v \in \mathbb{Z}_+$. From $zw = xy$ and $y = wv$ it follows that $z = xv$, and likewise from $zw = xy$ and $w = xu$ it follows that $y = zu$. Thus $x \trianglelefteq z \trianglelefteq y$.

(ii) Follows by (i) by observing that $x = z$ if and only if $y = w$, which is equivalent to the claim that $x \neq z$ if and only if $y \neq w$. $\square$

Definition 5.16. Let $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. A function $h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is the *D-convolution* of the functions f and g if

$$\forall\, x, y \in \mathbb{Z}_+ : h(x, y) = \sum_{\substack{x \trianglelefteq z \trianglelefteq y \\ zw = xy}} f(x, z) g(x, w),$$

where the sum runs over all ordered pairs $\langle z, w \rangle$ of positive integers satisfying $x \trianglelefteq z \trianglelefteq y$ and $zw = xy$.

Remark. The letter 'D' in D-convolution refers to 'divisibility ordering'.

Remark. If $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ and $x, y \in \mathbb{Z}_+$ are such that $x \ntrianglelefteq y$, then by Lemma 5.3

$$\sum_{\substack{x \trianglelefteq z \trianglelefteq y \\ zw = xy}} f(x,z)g(x,w) = \sum_{z,w\,\in\,\emptyset} f(x,z)g(x,w) = 0.$$

Thus, if the function h is the D-convolution of f and g, then $h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$.

Lemma 5.4. *The following property holds in the factor lattice* $(\mathbb{Z}_+, \trianglelefteq)$*. If* $x, y \in \mathbb{Z}_+$ *are such that* $x \trianglelefteq y$*, then the function*

$$\beta : [\,x, y\,] \to [\,x, y\,] : \beta(z) = xy/z$$

is one-to-one (injective) and onto (surjective).

Proof. Let $x, y \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$. If $z \in \mathbb{Z}_+$ is such that $z \in [\,x, y\,]$, then by Lemma 5.2 there exists $w \in \mathbb{Z}_+$ such that $w \in [\,x, y\,]$ and $w = xy/z$, and thus

$$\beta : [\,x, y\,] \to [\,x, y\,] : \beta(z) = xy/z$$

is a function from $[\,x, y\,]$ into $[\,x, y\,]$. Let $z_1, z_2 \in [\,x, y\,]$ be such that $\beta(z_1) = \beta(z_2)$. Since $\beta(z_1) = xy/z_1$, it follows that $z_1 \cdot \beta(z_1) = xy$, and likewise, since $\beta(z_2) = xy/z_2$, it follows that $z_2 \cdot \beta(z_2) = xy$. Thus $z_1 \cdot \beta(z_1) = z_2 \cdot \beta(z_2)$, and therefore $z_1 = z_2$. Thus the function β is one-to-one function (injective) in the set $[\,x, y\,]$, and therefore it is also onto (surjective). □

Lemma 5.5. *If* $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$*, then*

$$\sum_{\substack{x \trianglelefteq z \trianglelefteq y \\ zw = xy}} f(x,z)g(x,w) = \sum_{\substack{x \trianglelefteq z \trianglelefteq y \\ zw = xy}} f(x,w)g(x,z).$$

Proof. Follows by Lemma 5.4. □

In general, the role change of variables demonstrated by Lemma 5.5 can be applied to any summation over the conditions $x \trianglelefteq z \trianglelefteq y$ and $zw = xy$.

Definition 5.17. The binary operation $*_D$ in the set $\mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, i.e. the *D-convolution of arithmetic incidence functions*, is defined as follows:

$$*_D : \mathbb{I}[\mathbb{Z}_+, \trianglelefteq] \times \mathbb{I}[\mathbb{Z}_+, \trianglelefteq] \to \mathbb{I}[\mathbb{Z}_+, \trianglelefteq] : *_D\,(f, g) = h,$$

where the function h is the D-convolution of the functions f and g, denoted by $f *_D g$.

For reasons of clarity, it is in order to adopt a simplified version of the introduced notation for the D-convolution, and use the notation $f * g$ instead of $f *_D g$. The use of the notation $f * g$ is backed up by the following theorems which present some of the basic properties of the D-convolution of arithmetic incidence functions reflecting its close relationship with the Dirichlet convolution of arithmetic functions.

Definition 5.18. A function $F \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is the *divisibility order summatory function* of a function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ if

$$\forall\, x, y \in \mathbb{Z}_+ : F(x, y) = \sum_{x \trianglelefteq z \trianglelefteq y} f(x, z).$$

Theorem 5.15. *Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. Then $F \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is the divisibility order summatory function of the function f if and only if*

$$F = f * \zeta, \quad \textit{where } \zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq].$$

Proof. Follows by Definitions 5.18 and 5.17. $\square$

Let us proceed by showing that the D-convolution of arithmetic incidence functions shares all the basic properties of the Dirichlet convolution of arithmetic functions.

Lemma 5.6. *Let $x, y, w, u, v \in \mathbb{Z}_+$. Then*

$$x \trianglelefteq uv/x \trianglelefteq y, \quad (uv/x)w = xy, \quad x \trianglelefteq u \trianglelefteq uv/x$$

if and only if

$$x \trianglelefteq u \trianglelefteq y, \quad u(vw/x) = xy, \quad x \trianglelefteq v \trianglelefteq vw/x.$$

Proof. Let $x, y, w, u, v \in \mathbb{Z}_+$. Let us first assume that

$$x \trianglelefteq uv/x \trianglelefteq y, \quad (uv/x)w = xy, \quad x \trianglelefteq u \trianglelefteq uv/x.$$

Since $u \trianglelefteq uv/x$ and $uv/x \trianglelefteq y$, it follows that $u \trianglelefteq y$. Thus (i) $x \trianglelefteq u \trianglelefteq y$. From $(uv/x)w = xy$ it follows that $uvw = x \cdot xy$. Since $u \trianglelefteq y$, it follows that $u \trianglelefteq xy$, and therefore $xy = ut$, where $t \in \mathbb{Z}_+$. Thus $uvw = xut$, and therefore $vw = xt$. Thus $t = vw/x$, and therefore (ii) $u(vw/x) = xy$. From $u \trianglelefteq uv/x$ it follows that $us = uv/x$, where $s \in \mathbb{Z}_+$. Thus $xus = uv$, and therefore $v = xs$. Thus $x \trianglelefteq v$. Since $x \trianglelefteq uv/x \trianglelefteq y$ and $(uv/x)w = xy$, it follows by Lemma 5.3 that $x \trianglelefteq w \trianglelefteq y$. From $x \trianglelefteq w$ it follows that $w = xr$, where $r \in \mathbb{Z}_+$. Since $vw = xt$, it follows that $vrw = xrt$. Thus $vrw = wt$, and therefore $t = vr$. Thus $v \trianglelefteq t$, and therefore $v \trianglelefteq vw/x$. Thus (iii) $x \trianglelefteq v \trianglelefteq vw/x$.

Let us next assume that

$$x \trianglelefteq u \trianglelefteq y, \quad u(vw/x) = xy, \quad x \trianglelefteq v \trianglelefteq vw/x.$$

From $x \trianglelefteq v$ it follows that $xt = v$, where $t \in \mathbb{Z}_+$. Thus $xut = uv$, and therefore $uv/x = ut$. Thus $u \trianglelefteq uv/x$, and therefore (iv) $x \trianglelefteq u \trianglelefteq uv/x$. From $u(vw/x) = xy$ it follows that $uvw = x \cdot xy$. Since $x \trianglelefteq u \trianglelefteq y$ and $u(vw/x) = xy$, it follows by Lemma 5.3 that $x \trianglelefteq vw/x \trianglelefteq y$, and therefore $y = (vw/x)s$, where $s \in \mathbb{Z}_+$. Thus $xy = vws$, and therefore $uvw = xvws$.

Thus $uv = xvs$, and therefore $uv/x = vs$. Thus $(uv/x)w = vws$, and therefore (v) $(uv/x)w = xy$. From $v \trianglelefteq vw/x$ it follows that $vw/x = vr$, where $r \in \mathbb{Z}_+$. Thus $vw = xvr$, and therefore $w = xr$. Since $xy = vws$ and $v = xt$, it follows that $y = wst$. Thus $x \trianglelefteq w \trianglelefteq y$. Since $x \trianglelefteq w \trianglelefteq y$ and $(uv/x)w = xy$, it follows by Lemma 5.3 that (vi) $x \trianglelefteq uv/x \trianglelefteq y$. □

Theorem 5.16. *The D-convolution of arithmetic incidence functions is associative.*

Proof. Let $f, g, h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ and $x, y \in \mathbb{Z}_+$. Then by Lemma 5.6

$$\begin{aligned}
[(f*g)*h](x,y) &= \sum_{\substack{x \trianglelefteq z \trianglelefteq y \\ zw=xy}} (f*g)(x,z)h(x,w) \\
&= \sum_{\substack{x \trianglelefteq z \trianglelefteq y \\ zw=xy}} \Big[\sum_{\substack{x \trianglelefteq u \trianglelefteq z \\ uv=xz}} f(x,u)g(x,v) \Big] h(x,w) \\
&= \sum_{\substack{x \trianglelefteq z \trianglelefteq y \\ zw=xy}} \sum_{\substack{x \trianglelefteq u \trianglelefteq z \\ uv=xz}} f(x,u)g(x,v)h(x,w) \\
&= \sum_{\substack{x \trianglelefteq z \trianglelefteq y \\ zw=xy \\ x \trianglelefteq u \trianglelefteq z \\ uv=xz}} f(x,u)g(x,v)h(x,w) \\
&= \sum_{\substack{x \trianglelefteq uv/x \trianglelefteq y \\ (uv/x)w=xy \\ x \trianglelefteq u \trianglelefteq uv/x \\ uv=x(uv/x)}} f(x,u)g(x,v)h(x,w) \\
&= \sum_{\substack{x \trianglelefteq u \trianglelefteq y \\ u(vw/x)=xy \\ x \trianglelefteq v \trianglelefteq vw/x \\ vw=x(vw/x)}} f(x,u)g(x,v)h(x,w) \\
&= \sum_{\substack{x \trianglelefteq u \trianglelefteq y \\ ut=xy \\ x \trianglelefteq v \trianglelefteq t \\ vw=xt}} f(x,u)g(x,v)h(x,w) \\
&= \sum_{\substack{x \trianglelefteq u \trianglelefteq y \\ ut=xy}} \sum_{\substack{x \trianglelefteq v \trianglelefteq t \\ vw=xt}} f(x,u)g(x,v)h(x,w) \\
&= \sum_{\substack{x \trianglelefteq u \trianglelefteq y \\ ut=xy}} \Big[f(x,u) \sum_{\substack{x \trianglelefteq v \trianglelefteq t \\ vw=xt}} g(x,v)h(x,w) \Big] \\
&= \sum_{\substack{x \trianglelefteq u \trianglelefteq y \\ ut=xy}} f(x,u)(g*h)(x,t) \\
&= [f*(g*h)](x,y).
\end{aligned}$$

Thus $(f*g)*h = f*(g*h)$, and therefore the D-convolution of arithmetic incidence functions is associative. □

Theorem 5.17. *The D-convolution of arithmetic incidence functions is commutative.*

Proof. Let $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ and $x, y \in \mathbb{Z}_+$. Then by Lemma 5.5

$$(f * g)(x,y) = \sum_{\substack{x \trianglelefteq z \trianglelefteq y \\ zw = xy}} f(x,z)g(x,w) = \sum_{\substack{x \trianglelefteq z \trianglelefteq y \\ zw = xy}} g(x,z)f(x,w) = (g * f)(x,y).$$

Thus $f * g = g * f$, and therefore the D-convolution of arithmetic incidence functions is commutative. □

Theorem 5.18. *The delta function $\delta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is the identity element with respect to the D-convolution of arithmetic incidence functions.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ and $x, y \in \mathbb{Z}_+$. Then

$$\begin{aligned}
(\delta * f)(x,y) &= \sum_{\substack{x \trianglelefteq z \trianglelefteq y \\ zw = xy}} \delta(x,z)f(x,w) \\
&= \delta(x,x)f(x,y) + \sum_{\substack{x \triangleleft z \trianglelefteq y \\ zw = xy}} \delta(x,z)f(x,w) \\
&= 1 \cdot f(x,y) + 0 \\
&= f(x,y).
\end{aligned}$$

Thus $\delta * f = f$. Since $\delta * f = f$, it follows by Theorem 5.17 that $f * \delta = f$. Thus $\delta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is the identity element with respect to the D-convolution of arithmetic incidence functions. □

Theorem 5.19. *The algebraic structure $\langle \mathbb{I}[\mathbb{Z}_+, \trianglelefteq], * \rangle$ is a commutative semigroup with identity element.*

Proof. Follows by Theorems 5.16, 5.17, and 5.18. □

If $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is invertible with respect to the D-convolution, then its D-convolution inverse is denoted by f^{*-1}. Since the delta function $\delta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is the identity element with respect to the D-convolution, it is invertible with respect to the D-convolution being its own inverse, i.e. $\delta^{*-1} = \delta$. Let us next introduce a necessary and sufficient condition for an arithmetic incidence function to have a D-convolution inverse.

Theorem 5.20. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ has a D-convolution inverse if and only if*

$$\forall\, x \in \mathbb{Z}_+ : f(x,x) \neq 0.$$

This D-convolution inverse f^{*-1} is defined recursively as follows:

$$f^{*-1}(x,y) = \begin{cases} f(x,y)^{-1} & \text{if } x = y, \\ -\, f(x,x)^{-1} \displaystyle\sum_{\substack{x \trianglelefteq z \triangleleft y \\ zw = xy}} f^{*-1}(x,z)f(x,w) & \text{if } x \triangleleft y, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. Let us first assume that f has a D-convolution inverse. Then $f * f^{*-1} = \delta$, where $f^{*-1} \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is the inverse of f. Let $x \in \mathbb{Z}_+$. Let us assume that $f(x,x) = 0$. Then

$$\begin{aligned}
\delta(x,x) &= (f * f^{*-1})(x,x) \\
&= \sum_{\substack{x \trianglelefteq z \trianglelefteq x \\ zw = xx}} f(x,z) f^{*-1}(x,w) \\
&= f(x,x) f^{*-1}(x,x) \\
&= 0 \cdot f^{*-1}(x,x) \\
&= 0.
\end{aligned}$$

This contradicts the fact that $\delta(x,x) = 1$, and therefore $f(x,x) \neq 0$. Thus

$$\forall\, x \in \mathbb{Z}_+ : f(x,x) \neq 0.$$

Let us next assume that

$$\forall\, x \in \mathbb{Z}_+ : f(x,x) \neq 0.$$

Let us define the function $g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ recursively as follows:

$$g(x,y) = \begin{cases} f(x,y)^{-1} & \text{if } x = y, \\ -f(x,x)^{-1} \displaystyle\sum_{\substack{x \trianglelefteq z \triangleleft y \\ zw = xy}} g(x,z) f(x,w) & \text{if } x \triangleleft y, \\ 0 & \text{otherwise.} \end{cases}$$

Let $x, y \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$. If $x = y$, then

$$(g * f)(x,y) = \sum_{\substack{x \trianglelefteq z \trianglelefteq x \\ zw = xy}} g(x,z) f(x,w) = g(x,x) f(x,x) = f(x,x)^{-1} f(x,x) = 1.$$

If $x \triangleleft y$, then by Lemma 5.3

$$\begin{aligned}
(g * f)(x,y) &= \sum_{\substack{x \trianglelefteq z \trianglelefteq y \\ zw = xy}} g(x,z) f(x,w) \\
&= \Big[\sum_{\substack{x \trianglelefteq z \triangleleft y \\ zw = xy}} g(x,z) f(x,w) \Big] + g(x,y) f(x,x) \\
&= \Big[\sum_{\substack{x \trianglelefteq z \triangleleft y \\ zw = xy}} g(x,z) f(x,w) \Big] + f(x,x) \Big[-f(x,x)^{-1} \sum_{\substack{x \trianglelefteq z \triangleleft y \\ zw = xy}} g(x,z) f(x,w) \Big] \\
&= \Big[\sum_{\substack{x \trianglelefteq z \triangleleft y \\ zw = xy}} g(x,z) f(x,w) \Big] + \Big[- \sum_{\substack{x \trianglelefteq z \triangleleft y \\ zw = xy}} g(x,z) f(x,w) \Big] \\
&= 0.
\end{aligned}$$

Thus $g * f = \delta$. Since $g * f = \delta$, it follows by Theorem 5.17 that $f * g = \delta$. Thus g is the D-convolution inverse of f, i.e. $g = f^{*-1}$. $\square$

Theorem 5.21. *Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be such that*

$$\forall\, x \in \mathbb{Z}_+ : f(x,x) = 1.$$

The D-convolution inverse f^{*-1} of the function f is defined recursively as follows:

$$f^{*-1}(x,y) = \begin{cases} 1 & \text{if } x = y, \\ -\displaystyle\sum_{\substack{x \trianglelefteq z \triangleleft y \\ zw = xy}} f^{*-1}(x,z) f(x,w) & \text{if } x \triangleleft y, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Follows by Theorem 5.20. □

Lemma 5.7. *If $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is such that*

$$\forall\, x \in \mathbb{Z}_+ : f(x,x) \neq 0,$$

then

$$\forall\, x \in \mathbb{Z}_+ : \forall\, p \in \mathbb{P} : \forall\, n \in \mathbb{N} : \\ f^{*-1}(xp^n, xp^{n+1}) = -f(x,x)^{-1} f^{*-1}(xp^n, xp^n) f(xp^n, xp^{n+1}).$$

Proof. Follows by Theorem 5.20. □

Theorem 5.22. *If $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ are invertible with respect to the D-convolution, then $f * g$ is invertible with respect to the D-convolution.*

Proof. Let $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be invertible with respect to the D-convolution, and let $x \in \mathbb{Z}_+$. Then by Theorem 5.20 $f(x,x) \neq 0$ and $g(x,x) \neq 0$. Since $(f * g)(x,x) = f(x,x)g(x,x)$, it follows that $(f * g)(x,x) \neq 0$. Thus by Theorem 5.20 $f * g$ is invertible with respect to the D-convolution. □

The following theorem presents the general inversion formula with respect to the D-convolution for $\mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$.

Theorem 5.23. *If $h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is invertible with respect to the D-convolution, then*

$$\forall\, f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq] : f = g * h \Leftrightarrow g = f * h^{*-1}.$$

Proof. Let $h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be invertible with respect to the D-convolution, and let $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. If $f = g * h$, then by Theorems 5.18 and 5.16

$$g = g * \delta = g * (h * h^{*-1}) = (g * h) * h^{*-1} = f * h^{*-1}.$$

If $g = f * h^{*-1}$, then by Theorems 5.18 and 5.16

$$f = f * \delta = f * (h^{*-1} * h) = (f * h^{*-1}) * h = g * h.$$

□

Theorem 5.24. *Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. If $F \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is the divisibility order summatory function of the function f, then*

$$f = F * \zeta^{*-1}, \quad \textit{where } \zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq].$$

Proof. Follows by Theorems 5.15 and 5.23. □

Theorem 5.25. *The D-convolution of arithmetic incidence functions distributes over the addition of arithmetic incidence functions.*

Proof. Let $f, g, h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ and $x, y \in \mathbb{Z}_+$. Then

$$\begin{aligned}
[f * (g+h)](x,y) &= \sum_{\substack{x \trianglelefteq z \trianglelefteq y \\ zw = xy}} f(x,z)(g+h)(x,w) \\
&= \sum_{\substack{x \trianglelefteq z \trianglelefteq y \\ zw = xy}} f(x,z)(g(x,w) + h(x,w)) \\
&= \sum_{\substack{x \trianglelefteq z \trianglelefteq y \\ zw = xy}} f(x,z)g(x,w) + f(x,z)h(x,w) \\
&= \sum_{\substack{x \trianglelefteq z \trianglelefteq y \\ zw = xy}} f(x,z)g(x,w) + \sum_{\substack{x \trianglelefteq z \trianglelefteq y \\ zw = xy}} f(x,z)h(x,w) \\
&= (f * g)(x,y) + (f * h)(x,y) \\
&= [(f * g) + (f * h)](x,y).
\end{aligned}$$

Thus $f * (g+h) = (f * g) + (f * h)$, and therefore the left distributive law holds. By Theorem 5.17 and by the left distributive law

$$(f+g) * h = h * (f+g) = (h * f) + (h * g) = (f * h) + (g * h),$$

and therefore the right distributive law holds. Thus the D-convolution of arithmetic incidence functions distributes over the addition of arithmetic incidence functions. □

As is usual for the addition and convolution, the addition of arithmetic incidence functions does not distribute over the D-convolution of arithmetic incidence functions. Consequently, following the usual convention, D-convolution is performed before addition in the absence of parentheses, and therefore the related distributive laws take the following form:

$$f * (g+h) = f * g + f * h \quad \text{and} \quad (f+g) * h = f * h + g * h.$$

Theorem 5.26. *The algebraic structure $\langle \mathbb{I}[\mathbb{Z}_+, \trianglelefteq], +, * \rangle$ is a commutative ring with unity.*

Proof. Follows by Theorems 5.6, 5.19, and 5.25. □

The multiplication of arithmetic incidence functions does not distribute over the D-convolution of arithmetic incidence functions. However, there are such functions in the set $\mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ for which this property holds. The following theorem presents a characterization of such functions.

Theorem 5.27. *Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. Then*

$$\forall\, g, h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq] : f(g * h) = (fg) * (fh)$$

if and only if

$$\forall\, x, y, z, w \in \mathbb{Z}_+ : (x \trianglelefteq z \trianglelefteq y \text{ and } zw = xy) \Rightarrow f(x, y) = f(x, z)f(x, w).$$

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. Let us first assume that

$$\forall\, g, h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq] : f(g * h) = (fg) * (fh).$$

Let $x, y, z, w \in \mathbb{Z}_+$ be such that $x \trianglelefteq z \trianglelefteq y$ and $zw = xy$. Let us define the functions $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ as follows:

$$g : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C} : g(u, v) = \begin{cases} 1 & \text{if } u = x \text{ and } v = z, \\ 0 & \text{otherwise,} \end{cases}$$
$$h : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C} : h(u, v) = \begin{cases} 1 & \text{if } u = x \text{ and } v = w, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} [f(g * h)](x, y) &= f(x, y)(g * h)(x, y) \\ &= f(x, y) \sum_{\substack{x \trianglelefteq s \trianglelefteq y \\ st = xy}} g(x, s)h(x, t) \\ &= f(x, y)[g(x, z)h(x, w)] \\ &= f(x, y), \end{aligned}$$

and

$$\begin{aligned} [(fg) * (fh)](x, y) &= \sum_{\substack{x \trianglelefteq s \trianglelefteq y \\ st = xy}} (fg)(x, s)(fh)(x, t) \\ &= \sum_{\substack{x \trianglelefteq s \trianglelefteq y \\ st = xy}} f(x, s)g(x, s)f(x, t)h(x, t) \\ &= f(x, z)g(x, z)f(x, w)h(x, w) \\ &= f(x, z)f(x, w). \end{aligned}$$

By the assumption $f(g * h) = (fg) * (fh)$, and therefore

$$f(x, y) = f(x, z)f(x, w).$$

Let us next assume that

$$\forall\, x, y, z, w \in \mathbb{Z}_+ : (x \trianglelefteq z \trianglelefteq y \text{ and } zw = xy) \Rightarrow f(x, y) = f(x, z)f(x, w).$$

Let $g, h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, and let $x, y \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$. Then by the assumption

$$\begin{aligned}
[f(g * h)](x, y) &= f(x, y)(g * h)(x, y) \\
&= f(x, y) \sum_{\substack{x \trianglelefteq z \trianglelefteq y \\ zw = xy}} g(x, z)h(x, w) \\
&= \sum_{\substack{x \trianglelefteq z \trianglelefteq y \\ zw = xy}} f(x, y)g(x, z)h(x, w) \\
&= \sum_{\substack{x \trianglelefteq z \trianglelefteq y \\ zw = xy}} f(x, z)f(x, w)g(x, z)h(x, w) \\
&= \sum_{\substack{x \trianglelefteq z \trianglelefteq y \\ zw = xy}} (fg)(x, z)(fh)(x, w) \\
&= [(fg) * (fh)](x, y).
\end{aligned}$$

Thus $f(g * h) = (fg) * (fh)$. □

The following lemmas prove to be very useful when showing that a specific property of an arithmetic incidence function is closed under the D-convolution and the D-convolution inverses.

Lemma 5.8. *Let $x, y, z, w \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$, $z \trianglelefteq w$, and $\gcf(x, z) = \gcf(y, w)$. If $u, v \in \mathbb{Z}_+$ are such that $x \trianglelefteq u \trianglelefteq y$ and $z \trianglelefteq v \trianglelefteq w$, then $\gcf(x, z) = \gcf(u, v)$.*

Proof. Elementary. □

Lemma 5.9. *Let $(\mathbb{Z}_+, \trianglelefteq)$ be the factor lattice. If $x, y, z, w \in \mathbb{Z}_+$ are such that $x \trianglelefteq y$, $z \trianglelefteq w$, and $\gcf(x, z) = \gcf(y, w)$, then the functions*

$$\begin{aligned}
&\beta_1 : [\,x, y\,] \times [\,z, w\,] \to [\,\operatorname{lcm}(x, z), \operatorname{lcm}(y, w)\,] : \beta_1(u, v) = \operatorname{lcm}(u, v), \\
&\beta_2 : [\,\operatorname{lcm}(x, z), \operatorname{lcm}(y, w)\,] \to [\,x, y\,] \times [\,z, w\,] : \beta_2(t) = \langle \gcf(t, y), \gcf(t, w) \rangle
\end{aligned}$$

are both one-to-one (injective) and onto (surjective).

Proof. Let $x, y, z, w \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$, $z \trianglelefteq w$, and $\gcf(x, z) = \gcf(y, w)$. Then the intervals $[\,x, y\,]$, $[\,z, w\,]$, and $[\,\operatorname{lcm}(x, z), \operatorname{lcm}(y, w)\,]$ are nonempty.

Let $\langle u, v \rangle \in [\,x, y\,] \times [\,z, w\,]$. Then $x \trianglelefteq u \trianglelefteq y$ and $z \trianglelefteq v \trianglelefteq w$, and therefore $\operatorname{lcm}(x, z) \trianglelefteq \operatorname{lcm}(u, v) \trianglelefteq \operatorname{lcm}(y, w)$, i.e. $\operatorname{lcm}(u, v) \in [\,\operatorname{lcm}(x, z), \operatorname{lcm}(y, w)\,]$. Thus

$$\beta_1 : [\,x, y\,] \times [\,z, w\,] \to [\,\operatorname{lcm}(x, z), \operatorname{lcm}(y, w)\,] : \beta_1(u, v) = \operatorname{lcm}(u, v)$$

is a function.

Let $t \in [\,\mathrm{lcm}(x,z), \mathrm{lcm}(y,w)\,]$. Then $\mathrm{lcm}(x,z) \trianglelefteq t \trianglelefteq \mathrm{lcm}(y,w)$. Thus

$$x = \mathrm{gcf}(\mathrm{lcm}(x,z), x) \trianglelefteq \mathrm{gcf}(t,y) \trianglelefteq y$$

and

$$z = \mathrm{gcf}(\mathrm{lcm}(x,z), z) \trianglelefteq \mathrm{gcf}(t,w) \trianglelefteq w,$$

and therefore $\langle \mathrm{gcf}(t,y), \mathrm{gcf}(t,w)\rangle \in [\,x,y\,] \times [\,z,w\,]$. Thus

$$\beta_2 : [\,\mathrm{lcm}(x,z), \mathrm{lcm}(y,w)\,] \to [\,x,y\,] \times [\,z,w\,] : \beta_2(t) = \langle \mathrm{gcf}(t,y), \mathrm{gcf}(t,w)\rangle$$

is a function.

Let $\langle u,v\rangle \in [\,x,y\,] \times [\,z,w\,]$. Then $x \trianglelefteq u \trianglelefteq y$ and $z \trianglelefteq v \trianglelefteq w$. Thus by Lemma 5.8 $\mathrm{gcf}(x,z) = \mathrm{gcf}(u,w)$ and $\mathrm{gcf}(x,z) = \mathrm{gcf}(y,v)$, and therefore $\mathrm{gcf}(u,w) = \mathrm{gcf}(v,y)$. Thus by Theorems 2.17 and 2.19

$$\begin{aligned}
\beta_2(\beta_1(u,v)) &= \beta_2(\mathrm{lcm}(u,v)) \\
&= \langle \mathrm{gcf}(\mathrm{lcm}(u,v),y), \mathrm{gcf}(\mathrm{lcm}(u,v),w)\rangle \\
&= \langle \mathrm{gcf}(\mathrm{lcm}(u,v),y), \mathrm{gcf}(\mathrm{lcm}(v,u),w)\rangle \\
&= \langle \mathrm{lcm}(u,\mathrm{gcf}(v,y)), \mathrm{lcm}(v,\mathrm{gcf}(u,w))\rangle \\
&= \langle \mathrm{lcm}(u,\mathrm{gcf}(u,w)), \mathrm{lcm}(v,\mathrm{gcf}(v,y))\rangle \\
&= \langle u,v\rangle\,.
\end{aligned}$$

Thus

$$\forall\,\langle u,v\rangle \in [\,x,y\,] \times [\,z,w\,] : \beta_2(\beta_1(u,v)) = \langle u,v\rangle\,,$$

and therefore β_1 is one-to-one and β_2 is onto.

Let $t \in [\,\mathrm{lcm}(x,z), \mathrm{lcm}(y,w)\,]$. Then $\mathrm{lcm}(x,z) \trianglelefteq t \trianglelefteq \mathrm{lcm}(y,w)$, and therefore by Theorem 2.18

$$\begin{aligned}
\beta_1(\beta_2(t)) &= \beta_1(\mathrm{gcf}(t,y), \mathrm{gcf}(t,w)) \\
&= \mathrm{lcm}(\mathrm{gcf}(t,y), \mathrm{gcf}(t,w)) \\
&= \mathrm{gcf}(t, \mathrm{lcm}(y,w)) \\
&= t.
\end{aligned}$$

Thus

$$\forall\, t \in [\,\mathrm{lcm}(x,z), \mathrm{lcm}(y,w)\,] : \beta_1(\beta_2(t)) = t,$$

and therefore β_2 is one-to-one and β_1 is onto. $\square$

Remark. The property of the factor lattice $(\mathbb{Z}_+, \trianglelefteq)$ introduced by Lemma 5.9 is, in effect, a specific application of a common property shared by all locally distributive local lattices (see [47, p. 623], [25, p. 321]).

Lemma 5.10. *Let $x, y, z, w \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$, $z \trianglelefteq w$, and* $\gcf(x, z) = \gcf(y, w)$. *If $u, v, s, t \in \mathbb{Z}_+$ are such that $x \trianglelefteq u \trianglelefteq y$, $z \trianglelefteq s \trianglelefteq w$, $x \trianglelefteq v \trianglelefteq y$ and $z \trianglelefteq t \trianglelefteq w$, then*

$$\operatorname{lcm}(u, s) \operatorname{lcm}(v, t) = \operatorname{lcm}(x, z) \operatorname{lcm}(y, w)$$

if and only if

$$uv = xy \text{ and } st = zw.$$

Proof. Let $x, y, z, w \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$, $z \trianglelefteq w$, and $\operatorname{gcf}(x, z) = \operatorname{gcf}(y, w)$, and let $u, v, s, t \in \mathbb{Z}_+$ be such that $x \trianglelefteq u \trianglelefteq y$, $z \trianglelefteq s \trianglelefteq w$, $x \trianglelefteq v \trianglelefteq y$ and $z \trianglelefteq t \trianglelefteq w$. Let us first assume that $\operatorname{lcm}(u, s) \operatorname{lcm}(v, t) = \operatorname{lcm}(x, z) \operatorname{lcm}(y, w)$. Since $x \trianglelefteq u \trianglelefteq y$ and $z \trianglelefteq s \trianglelefteq w$, it follows that $u \trianglelefteq xy$ and $s \trianglelefteq zw$, and therefore $xy = uq$ and $zw = sr$, where $q, r \in \mathbb{Z}_+$. Thus by Lemma 5.3 $x \trianglelefteq q \trianglelefteq y$ and $z \trianglelefteq r \trianglelefteq w$. Let us note that by Lemma 5.8 $\operatorname{gcf}(x, z) = \operatorname{gcf}(u, s)$ and $\operatorname{gcf}(y, w) = \operatorname{gcf}(q, r)$. Thus

$$\begin{aligned}
\operatorname{gcf}(x, z) \operatorname{lcm}(u, s) \operatorname{gcf}(y, w) \operatorname{lcm}(q, r) &= \operatorname{gcf}(u, s) \operatorname{lcm}(u, s) \operatorname{gcf}(q, r) \operatorname{lcm}(q, r) \\
&= usqr \\
&= uqsr \\
&= xyzw \\
&= xzyw \\
&= \operatorname{gcf}(x, z) \operatorname{lcm}(x, z) \operatorname{gcf}(y, w) \operatorname{lcm}(y, w),
\end{aligned}$$

and therefore $\operatorname{lcm}(u, s) \operatorname{lcm}(q, r) = \operatorname{lcm}(x, z) \operatorname{lcm}(y, w)$. Thus by the assumption $\operatorname{lcm}(q, r) = \operatorname{lcm}(v, t)$. Let us note that by Lemma 5.9 the function

$$\beta_1 : [\, x, y \,] \times [\, z, w \,] \to [\, \operatorname{lcm}(x, z), \operatorname{lcm}(y, w) \,] : \beta_1(u, v) = \operatorname{lcm}(u, v)$$

is one-to-one, and therefore $q = v$ and $r = t$. Thus $uv = xy$ and $st = zw$.

Let us next assume that $uv = xy$ and $st = zw$. Let us note that by Lemma 5.8 $\operatorname{gcf}(x, z) = \operatorname{gcf}(u, s)$ and $\operatorname{gcf}(y, w) = \operatorname{gcf}(v, t)$. Thus

$$\begin{aligned}
\operatorname{gcf}(x, z) \operatorname{lcm}(u, s) \operatorname{gcf}(y, w) \operatorname{lcm}(v, t) &= \operatorname{gcf}(u, s) \operatorname{lcm}(u, s) \operatorname{gcf}(v, t) \operatorname{lcm}(v, t) \\
&= usvt \\
&= uvst \\
&= xyzw \\
&= xzyw \\
&= \operatorname{gcf}(x, z) \operatorname{lcm}(x, z) \operatorname{gcf}(y, w) \operatorname{lcm}(y, w),
\end{aligned}$$

and therefore $\operatorname{lcm}(u, s) \operatorname{lcm}(v, t) = \operatorname{lcm}(x, z) \operatorname{lcm}(y, w)$. $\square$

5.6 The Möbius Function

By Theorem 5.20 the zeta function $\zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ has an inverse with respect to the D-convolution. This D-convolution inverse of the zeta function, being a generalization of the Dirichlet convolution inverse of the zeta function of arithmetic functions, is referred to as the *Möbius function*.

Definition 5.19. The *Möbius function* $\mu \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is defined as follows:

$$\mu = \zeta^{*-1}.$$

Theorem 5.28. *The delta function $\delta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is the divisibility order summatory function of the Möbius function $\mu \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, i.e.*

$$\sum_{x \trianglelefteq z \trianglelefteq y} \mu(x, z) = \begin{cases} 1 & \textit{if } x = y, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Follows by Theorem 5.15 from the fact that $\mu * \zeta = \delta$. □

Theorem 5.29. *The Möbius function $\mu \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is determined recursively as follows:*

$$\mu(x, y) = \begin{cases} 1 & \textit{if } x = y, \\ -1 & \textit{if } x \blacktriangleleft y, \\ -\sum\limits_{x \trianglelefteq z \triangleleft y} \mu(x, z), & \textit{if } x \triangleleft y, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Let $x, y \in \mathbb{Z}_+$ be such that $x \blacktriangleleft y$. Then $x \triangleleft y$ and

$$\forall\, z \in \mathbb{Z}_+ : (x \trianglelefteq z \text{ and } z \triangleleft y) \Rightarrow z = x,$$

and therefore

$$-\sum_{x \trianglelefteq z \triangleleft y} \mu(x, z) = -\mu(x, x).$$

Thus by Theorem 5.21 and Lemma 5.4

$$\mu(x, y) = \begin{cases} 1 & \text{if } x = y, \\ -1 & \text{if } x \blacktriangleleft y, \\ -\sum\limits_{x \trianglelefteq z \triangleleft y} \mu(x, z) & \text{if } x \triangleleft y, \\ 0 & \text{otherwise.} \end{cases}$$

□

Lemma 5.11. *The factor relation $\trianglelefteq$ in the set $\mathbb{Z}_+$ satisfies the following:*

$$\forall\, x, z \in \mathbb{Z}_+ : x \blacktriangleleft xz \Leftrightarrow z \in \mathbb{P}.$$

Proof. Elementary. $\square$

Lemma 5.12. *The factor relation $\trianglelefteq$ in the set $\mathbb{Z}_+$ satisfies the following:*

$$\forall\, x, z \in \mathbb{Z}_+ : \forall\, p \in \mathbb{P} : \forall\, m, n \in \mathbb{N} : xp^n \trianglelefteq z \blacktriangleleft xp^{n+m+1} \Rightarrow z = xp^{n+m}.$$

Proof. Elementary. $\square$

Lemma 5.13. *The factor relation $\trianglelefteq$ in the set $\mathbb{Z}_+$ satisfies the following:*

$$\begin{aligned} &\forall\, x, z \in \mathbb{Z}_+ : \forall\, p \in \mathbb{P} : \forall\, m, n \in \mathbb{N} : \left[xp^n \trianglelefteq z \trianglelefteq xp^{n+m}\right] \\ &\qquad \Rightarrow \exists\, k \in \mathbb{N} : n \leq n + k \leq n + m \text{ and } z = xp^{n+k}. \end{aligned}$$

Proof. Elementary. $\square$

Theorem 5.30. *The following holds for the Möbius function $\mu \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$:*

(i) $\forall\, x \in \mathbb{Z}_+ : \forall\, p \in \mathbb{P} : \forall\, n \in \mathbb{N} : \mu(xp^n, xp^{n+1}) = -1$,

(ii) $\forall\, x \in \mathbb{Z}_+ : \forall\, p \in \mathbb{P} : \forall\, m, n \in \mathbb{N} : \mu(xp^n, xp^{n+m+2}) = 0$.

Proof. (i) Follows by Lemma 5.7.

(ii) Let $x \in \mathbb{Z}_+$, $p \in \mathbb{P}$, and $m, n \in \mathbb{N}$. Since by Lemma 5.11

$$xp^n \triangleleft xp^{n+m+1} \blacktriangleleft xp^{n+m+2}$$

and by Lemma 5.12

$$\forall\, z \in \mathbb{Z}_+ : xp^n \trianglelefteq z \blacktriangleleft xp^{n+m+2} \Rightarrow z = xp^{n+m+1},$$

it follows by Theorem 5.29 and Lemma 5.13 that

$$\begin{aligned} \mu(xp^n, xp^{n+m+2}) &= -\sum_{xp^n \trianglelefteq z \triangleleft xp^{n+m+2}} \mu(xp^n, z) \\ &= -\sum_{xp^n \trianglelefteq z \trianglelefteq xp^{n+m+1}} \mu(xp^n, z) \\ &= \Big[-\sum_{xp^n \trianglelefteq z \triangleleft xp^{n+m+1}} \mu(xp^n, z)\Big] + \Big[-\mu(xp^n, xp^{n+m+1})\Big] \\ &= \mu(xp^n, xp^{n+m+1}) + \Big[-\mu(xp^n, xp^{n+m+1})\Big] \\ &= 0. \end{aligned}$$

$\square$

Lemma 5.14. *Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be invertible with respect to the D-convolution. If $f^{*-1} = f\mu$, where $\mu \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, then*

$$\forall\, x \in \mathbb{Z}_+ : \forall\, p \in \mathbb{P} : \forall\, n \in \mathbb{N} : f(x,x)f(x,xp^{n+1}) = f(x,xp)f(x,xp^n).$$

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be invertible with respect to the D-convolution, and let $f^{*-1} = f\mu$, where $\mu \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. Let $x \in \mathbb{Z}_+$, $p \in \mathbb{P}$, and $n \in \mathbb{N}$. Then by Theorem 5.30 and Lemma 5.13

$$\begin{aligned}
0 &= \delta(x, xp^{n+1}) \\
&= [(f\mu) * f](x, xp^{n+1}) \\
&= \sum_{\substack{x \trianglelefteq z \trianglelefteq xp^{n+1} \\ zw = x \cdot xp^{n+1}}} (f\mu)(x,z)f(x,w) \\
&= \sum_{\substack{x \trianglelefteq z \trianglelefteq xp^{n+1} \\ zw = x \cdot xp^{n+1}}} f(x,z)\mu(x,z)f(x,w) \\
&= f(x,x)\mu(x,x)f(x,xp^{n+1}) + f(x,xp)\mu(x,xp)f(x,xp^n) \\
&= f(x,x)f(x,xp^{n+1}) + \Big[-f(x,xp)f(x,xp^n)\Big],
\end{aligned}$$

and therefore $f(x,x)f(x,xp^{n+1}) = f(x,xp)f(x,xp^n)$. ☐

Theorem 5.31. *The Möbius inversion formula for $\mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is the following:*

$$\forall\, f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq] : f = g * \zeta \Leftrightarrow g = f * \mu.$$

Proof. Follows by Theorem 5.23. ☐

The following theorem presents the Möbius inversion formula, introduced in Theorem 5.31, in a different form.

Theorem 5.32. *The following holds for every $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$:*

$$\forall\, x, y \in \mathbb{Z}_+ : f(x,y) = \sum_{x \trianglelefteq z \trianglelefteq y} g(x,z)$$

if and only if

$$\forall\, x, y \in \mathbb{Z}_+ : g(x,y) = \sum_{\substack{x \trianglelefteq z \trianglelefteq y \\ zw = xy}} f(x,z)\mu(x,w).$$

Proof. Follows by Theorem 5.31 and Lemma 5.4. ☐

5.7 C-convolution

As the D-convolution of arithmetic incidence functions serves as a generalization of the Dirichlet convolution of arithmetic functions, it is natural to call for a corresponding generalization of the unitary convolution of arithmetic functions. For this end, let us introduce the *C-convolution* of arithmetic incidence functions.

Lemma 5.15. *Let $x, y, z, w \in \mathbb{Z}_+$. Then*

$$x \trianglelefteq z \trianglelefteq y, \quad zw = xy, \quad \mathrm{gcf}(z, w) = x, \quad \mathrm{lcm}(z, w) = y$$

if and only if

$$\mathrm{gcf}(z, w) = x, \quad \mathrm{lcm}(z, w) = y.$$

Proof. Let $x, y, z, w \in \mathbb{Z}_+$. Let us first assume that

$$x \trianglelefteq z \trianglelefteq y, \quad zw = xy, \quad \mathrm{gcf}(z, w) = x, \quad \mathrm{lcm}(z, w) = y.$$

Then $\mathrm{gcf}(z, w) = x$ and $\mathrm{lcm}(z, w) = y$.

Let us next assume that $\mathrm{gcf}(z, w) = x$ and $\mathrm{lcm}(z, w) = y$. Then $x \trianglelefteq z$, $z \trianglelefteq y$, and

$$zw = \mathrm{gcf}(z, w) \,\mathrm{lcm}(z, w) = xy.$$

Thus $x \trianglelefteq z \trianglelefteq y$, $zw = xy$, $\mathrm{gcf}(z, w) = x$, and $\mathrm{lcm}(z, w) = y$. □

Definition 5.20. Let $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. A function $h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is the *C-convolution* of the functions f and g if

$$\forall\, x, y \in \mathbb{Z}_+ : h(x, y) = \sum_{\substack{\mathrm{gcf}(z,w)=x \\ \mathrm{lcm}(z,w)=y}} f(x, z)g(x, w),$$

where the sum runs over all ordered pairs $\langle z, w\rangle$ of positive integers satisfying $\mathrm{gcf}(z, w) = x$ and $\mathrm{lcm}(z, w) = y$.

Remark. The letter 'C' in C-convolution refers to 'complementary', which, in turn, is motivated by Definition 2.23.

Remark. If $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ and $x, y \in \mathbb{Z}_+$ are such that $x \not\trianglelefteq y$, then by Lemmas 5.15 and 5.3

$$\sum_{\substack{\mathrm{gcf}(z,w)=x \\ \mathrm{lcm}(z,w)=y}} f(x, z)g(x, w) = \sum_{\substack{x \trianglelefteq z \trianglelefteq y \\ zw=xy \\ \mathrm{gcf}(z,w)=x \\ \mathrm{lcm}(z,w)=y}} f(x, z)g(x, w) = \sum_{z,w\, \in\, \emptyset} f(x, z)g(x, w) = 0.$$

Thus, if the function h is the C-convolution of f and g, then $h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$.

Definition 5.21. The binary operation $*_C$ in the set $\mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, i.e. the *$C$-convolution of arithmetic incidence functions*, is defined as follows:

$$*_C : \mathbb{I}[\mathbb{Z}_+, \trianglelefteq] \times \mathbb{I}[\mathbb{Z}_+, \trianglelefteq] \to \mathbb{I}[\mathbb{Z}_+, \trianglelefteq] : *_C\,(f, g) = h,$$

where the function h is the C-convolution of the functions f and g.

If $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ and $x, y \in \mathbb{Z}_+$ are such that $x \trianglelefteq y$, then by Lemma 5.15

$$(f *_C g)(x,y) = \sum_{\substack{\mathrm{gcf}(z,w)=x \\ \mathrm{lcm}(z,w)=y}} f(x,z)g(x,w) = \sum_{\substack{x \trianglelefteq z \trianglelefteq y \\ zw=xy \\ \mathrm{gcf}(z,w)=x \\ \mathrm{lcm}(z,w)=y}} f(x,z)g(x,w).$$

In light of the above observation the C-convolution is, in a sense, a restriction of the D-convolution. The difference between these two convolutions is that in the D-convolution every element of the interval $[\,x, y\,]$ is relevant in all cases, whereas in the C-convolution this is the case if and only if the interval $[\,x, y\,]$ is a Boolean lattice (see Definition 2.25). In effect, if $x, y \in \mathbb{Z}_+$ are such that the interval $[\,x, y\,]$ is a Boolean lattice, then the D-convolution and the C-convolution coincide for these x and y, i.e.

$$\forall\, f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq] : (f * g)(x,y) = (f *_C g)(x,y).$$

Definition 5.22. A function F is the *complementary summatory function* of a function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ if

$$\forall\, x, y \in \mathbb{Z}_+ : F(x,y) = \sum_{\substack{\mathrm{gcf}(z,w)=x \\ \mathrm{lcm}(z,w)=y}} f(x,z).$$

Theorem 5.33. *Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. Then $F \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is the complementary summatory function of the function f if and only if*

$$F = f *_C \zeta, \quad \textit{where } \zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq].$$

Proof. Follows by Definitions 5.22 and 5.21. □

The close relationship between the C-convolution and the D-convolution suggests that the C-convolution shares the basic properties of the D-convolution.

Lemma 5.16. *Let $x, y, z, w \in \mathbb{Z}_+$ be such that $zw = xy$. Then* $\mathrm{gcf}(z,w) = x$ *if and only if* $\mathrm{lcm}(z,w) = y$.

Proof. Elementary. □

Lemma 5.17. *Let $x, y, w, u, v \in \mathbb{Z}_+$. Then*

$$\mathrm{gcf}(\mathrm{lcm}(u,v),w) = x, \quad \mathrm{lcm}(\mathrm{lcm}(u,v),w) = y, \quad \mathrm{gcf}(u,v) = x$$

if and only if

$$\mathrm{gcf}(u,\mathrm{lcm}(v,w)) = x, \quad \mathrm{lcm}(u,\mathrm{lcm}(v,w)) = y, \quad \mathrm{gcf}(v,w) = x.$$

Proof. Let $x, y, w, u, v \in \mathbb{Z}_+$. Let us first assume that

$$\operatorname{gcf}(\operatorname{lcm}(u,v),w) = x, \quad \operatorname{lcm}(\operatorname{lcm}(u,v),w) = y, \quad \operatorname{gcf}(u,v) = x.$$

Then $x \trianglelefteq v$ and $x \trianglelefteq w$, and therefore by Theorem 2.14 $x \trianglelefteq \operatorname{gcf}(v,w)$. Since $v \trianglelefteq \operatorname{lcm}(u,v)$ and $w \trianglelefteq w$, it follows that $\operatorname{gcf}(v,w) \trianglelefteq \operatorname{gcf}(\operatorname{lcm}(u,v),w)$. Thus $\operatorname{gcf}(v,w) \trianglelefteq x$ and $x \trianglelefteq \operatorname{gcf}(v,w)$, and therefore (i) $\operatorname{gcf}(v,w) = x$. By Theorem 2.17

$$\operatorname{lcm}(u,\operatorname{lcm}(v,w)) = \operatorname{lcm}(\operatorname{lcm}(u,v),w),$$

and therefore (ii) $\operatorname{lcm}(u,\operatorname{lcm}(v,w)) = y$. Since $\operatorname{lcm}(u,\operatorname{lcm}(v,w)) = y$, it follows that $u \trianglelefteq y$. Thus $u \trianglelefteq xy$, and therefore $xy = ut$, where $t \in \mathbb{Z}_+$. On the other hand,

$$\begin{aligned} xy &= \operatorname{gcf}(\operatorname{lcm}(u,v),w)\operatorname{lcm}(\operatorname{lcm}(u,v),w) \\ &= \operatorname{lcm}(u,v)w \\ &= (uv/\operatorname{gcf}(u,v))w \\ &= (uv/x)w. \end{aligned}$$

Thus $x \cdot xy = uvw$, and therefore $xut = uvw$. Thus $vw = xt$, and therefore by (i) $t = \operatorname{lcm}(v,w)$. Thus $u\operatorname{lcm}(v,w) = xy$ and $\operatorname{lcm}(u,\operatorname{lcm}(v,w)) = y$, and therefore by Lemma 5.16 (iii) $\operatorname{gcf}(u,\operatorname{lcm}(v,w)) = x$.

Let us next assume that

$$\operatorname{gcf}(u,\operatorname{lcm}(v,w)) = x, \quad \operatorname{lcm}(u,\operatorname{lcm}(v,w)) = y, \quad \operatorname{gcf}(v,w) = x.$$

Then $x \trianglelefteq u$ and $x \trianglelefteq v$, and therefore by Theorem 2.14 $x \trianglelefteq \operatorname{gcf}(u,v)$. Since $u \trianglelefteq u$ and $v \trianglelefteq \operatorname{lcm}(v,w)$, it follows that $\operatorname{gcf}(u,v) \trianglelefteq \operatorname{gcf}(u,\operatorname{lcm}(v,w))$. Thus $\operatorname{gcf}(u,v) \trianglelefteq x$ and $x \trianglelefteq \operatorname{gcf}(u,v)$, and therefore (iv) $\operatorname{gcf}(u,v) = x$. By Theorem 2.17

$$\operatorname{lcm}(\operatorname{lcm}(u,v),w) = \operatorname{lcm}(u,\operatorname{lcm}(v,w)),$$

and therefore (v) $\operatorname{lcm}(\operatorname{lcm}(u,v),w) = y$. Since $\operatorname{lcm}(\operatorname{lcm}(u,v),w) = y$, it follows that $w \trianglelefteq y$. Thus $w \trianglelefteq xy$, and therefore $xy = wt$, where $t \in \mathbb{Z}_+$. On the other hand,

$$\begin{aligned} xy &= \operatorname{gcf}(u,\operatorname{lcm}(v,w))\operatorname{lcm}(u,\operatorname{lcm}(v,w)) \\ &= u\operatorname{lcm}(v,w) \\ &= u(vw/\operatorname{gcf}(v,w)) \\ &= u(vw/x). \end{aligned}$$

Thus $x \cdot xy = uvw$, and therefore $xwt = uvw$. Thus $uv = xt$, and therefore by (iv) $t = \operatorname{lcm}(u,v)$. Thus $\operatorname{lcm}(u,v)w = xy$ and $\operatorname{lcm}(\operatorname{lcm}(u,v),w) = y$, and therefore by Lemma 5.16 (vi) $\operatorname{gcf}(\operatorname{lcm}(u,v),w) = x$.

$\square$

Theorem 5.34. *The C-convolution of arithmetic incidence functions is associative.*

Proof. Let $f, g, h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ and $x, y \in \mathbb{Z}_+$. Then by Lemma 5.17

$$\begin{aligned}
[(f *_C g) *_C h](x,y) &= \sum_{\substack{\text{gcf}(z,w)=x \\ \text{lcm}(z,w)=y}} (f *_C g)(x,z)h(x,w) \\
&= \sum_{\substack{\text{gcf}(z,w)=x \\ \text{lcm}(z,w)=y}} \Big[\sum_{\substack{\text{gcf}(u,v)=x \\ \text{lcm}(u,v)=z}} f(x,u)g(x,v) \Big] h(x,w) \\
&= \sum_{\substack{\text{gcf}(z,w)=x \\ \text{lcm}(z,w)=y}} \sum_{\substack{\text{gcf}(u,v)=x \\ \text{lcm}(u,v)=z}} f(x,u)g(x,v)h(x,w) \\
&= \sum_{\substack{\text{gcf}(z,w)=x \\ \text{lcm}(z,w)=y \\ \text{gcf}(u,v)=x \\ \text{lcm}(u,v)=z}} f(x,u)g(x,v)h(x,w) \\
&= \sum_{\substack{\text{gcf}(\text{lcm}(u,v),w)=x \\ \text{lcm}(\text{lcm}(u,v),w)=y \\ \text{gcf}(u,v)=x \\ \text{lcm}(u,v)=\text{lcm}(u,v)}} f(x,u)g(x,v)h(x,w) \\
&= \sum_{\substack{\text{gcf}(u,\text{lcm}(v,w))=x \\ \text{lcm}(u,\text{lcm}(v,w))=y \\ \text{gcf}(v,w)=x \\ \text{lcm}(v,w)=\text{lcm}(v,w)}} f(x,u)g(x,v)h(x,w) \\
&= \sum_{\substack{\text{gcf}(u,t)=x \\ \text{lcm}(u,t)=y \\ \text{gcf}(v,w)=x \\ \text{lcm}(v,w)=t}} f(x,u)g(x,v)h(x,w) \\
&= \sum_{\substack{\text{gcf}(u,t)=x \\ \text{lcm}(u,t)=y}} \sum_{\substack{\text{gcf}(v,w)=x \\ \text{lcm}(v,w)=t}} f(x,u)g(x,v)h(x,w) \\
&= \sum_{\substack{\text{gcf}(u,t)=x \\ \text{lcm}(u,t)=y}} \Big[f(x,u) \sum_{\substack{\text{gcf}(v,w)=x \\ \text{lcm}(v,w)=t}} g(x,v)h(x,w) \Big] \\
&= \sum_{\substack{\text{gcf}(u,t)=x \\ \text{lcm}(u,t)=y}} f(x,u)(g *_C h)(x,t) \\
&= [f *_C (g *_C h)](x,y).
\end{aligned}$$

Thus $(f *_C g) *_C h = f *_C (g *_C h)$, and therefore the C-convolution of arithmetic incidence functions is associative. □

Theorem 5.35. *The C-convolution of arithmetic incidence functions is commutative.*

Proof. Let $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ and $x, y \in \mathbb{Z}_+$. Then

$$\begin{aligned}(f *_C g)(x,y) &= \sum_{\substack{\mathrm{gcf}(z,w)=x \\ \mathrm{lcm}(z,w)=y}} f(x,z)g(x,w) \\ &= \sum_{\substack{\mathrm{gcf}(w,z)=x \\ \mathrm{lcm}(w,z)=y}} g(x,w)f(x,z) \\ &= (g *_C f)(x,y).\end{aligned}$$

Thus $f *_C g = g *_C f$, and therefore the C-convolution of arithmetic incidence functions is commutative. □

Theorem 5.36. *The function $\delta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is the identity element with respect to the C-convolution of arithmetic incidence functions.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ and $x, y \in \mathbb{Z}_+$. Then

$$\begin{aligned}(\delta *_C f)(x,y) &= \sum_{\substack{\mathrm{gcf}(z,w)=x \\ \mathrm{lcm}(z,w)=y}} \delta(x,z)f(x,w) \\ &= \delta(x,x)f(x,y) + \sum_{\substack{\mathrm{gcf}(z,w)=x \\ \mathrm{lcm}(z,w)=y \\ z\neq x}} \delta(x,z)f(x,w) \\ &= 1 \cdot f(x,y) + 0 \\ &= f(x,y).\end{aligned}$$

Thus $\delta *_C f = f$. Since $\delta *_C f = f$, it follows by Theorem 5.35 that $f *_C \delta = f$. Thus $\delta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is the identity element with respect to the C-convolution of arithmetic incidence functions. □

Theorem 5.37. *The algebraic structure $\langle \mathbb{I}[\mathbb{Z}_+, \trianglelefteq], *_C \rangle$ is a commutative semigroup with identity element.*

Proof. Follows by Theorems 5.34, 5.35, and 5.36. □

If $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is invertible with respect to the C-convolution, then its C-convolution inverse is denoted by $f^{*_C -1}$. Since the delta function $\delta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is the identity element with respect to the C-convolution, it is invertible with respect to the C-convolution being its own inverse, i.e. $\delta^{*_C -1} = \delta$. Let us next introduce a necessary and sufficient condition for an arithmetic incidence function to have a C-convolution inverse.

Theorem 5.38. *A function* $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ *has a* C*-convolution inverse if and only if*

$$\forall\, x \in \mathbb{Z}_+ : f(x,x) \neq 0.$$

This C*-convolution inverse* $f^{*_C -1}$ *is defined recursively as follows:*

$$f^{*_C -1}(x,y) = \begin{cases} f(x,y)^{-1} & \text{if } x = y, \\ -f(x,x)^{-1} \displaystyle\sum_{\substack{\gcd(z,w)=x \\ \operatorname{lcm}(z,w)=y \\ z \neq y}} f^{*_C -1}(x,z) f(x,w) & \text{if } x \lhd y, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. Let us first assume that f has a C-convolution inverse. Then $f *_C f^{*_C -1} = \delta$, where $f^{*_C -1} \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is the inverse of f. Let $x \in \mathbb{Z}_+$. Let us assume that $f(x,x) = 0$. Then

$$\begin{aligned} \delta(x,x) &= (f *_C f^{*_C -1})(x,x) \\ &= \sum_{\substack{\operatorname{gcf}(z,w)=x \\ \operatorname{lcm}(z,w)=x}} f(x,z) f^{*_C -1}(x,w) \\ &= f(x,x) f^{*_C -1}(x,x) \\ &= 0 \cdot f^{*_C -1}(x,x) \\ &= 0. \end{aligned}$$

This contradicts the fact that $\delta(x,x) = 1$, and therefore $f(x,x) \neq 0$. Thus

$$\forall\, x \in \mathbb{Z}_+ : f(x,x) \neq 0.$$

Let us next assume that

$$\forall\, x \in \mathbb{Z}_+ : f(x,x) \neq 0.$$

Let us define the function $g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ recursively as follows:

$$g(x,y) = \begin{cases} f(x,y)^{-1} & \text{if } x = y, \\ -f(x,x)^{-1} \displaystyle\sum_{\substack{\operatorname{gcf}(z,w)=x \\ \operatorname{lcm}(z,w)=y \\ z \neq y}} g(x,z) f(x,w) & \text{if } x \lhd y, \\ 0 & \text{otherwise.} \end{cases}$$

Let $x, y \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$. If $x = y$, then

$$(g *_C f)(x,y) = \sum_{\substack{\operatorname{gcf}(z,w)=x \\ \operatorname{lcm}(z,w)=y}} g(x,z) f(x,w) = g(x,x) f(x,x) = f(x,x)^{-1} f(x,x) = 1.$$

If $x \lhd y$, then by Lemmas 5.15 and 5.3

$$\begin{aligned}
(g *_C f)(x,y) &= \sum_{\substack{\mathrm{gcf}(z,w)=x\\ \mathrm{lcm}(z,w)=y}} g(x,z)f(x,w)\\
&= \Big[\sum_{\substack{\mathrm{gcf}(z,w)=x\\ \mathrm{lcm}(z,w)=y\\ z\neq y}} g(x,z)f(x,w)\Big] + g(x,y)f(x,x)\\
&= \Big[\sum_{\substack{\mathrm{gcf}(z,w)=x\\ \mathrm{lcm}(z,w)=y\\ z\neq y}} g(x,z)f(x,w)\Big]\\
&\qquad + f(x,x)\Big[-f(x,x)^{-1}\sum_{\substack{\mathrm{gcf}(z,w)=x\\ \mathrm{lcm}(z,w)=y\\ z\neq y}} g(x,z)f(x,w)\Big]\\
&= \Big[\sum_{\substack{\mathrm{gcf}(z,w)=x\\ \mathrm{lcm}(z,w)=y\\ z\neq y}} g(x,z)f(x,w)\Big] + \Big[-\sum_{\substack{\mathrm{gcf}(z,w)=x\\ \mathrm{lcm}(z,w)=y\\ z\neq y}} g(x,z)f(x,w)\Big]\\
&= 0.
\end{aligned}$$

Thus $g *_C f = \delta$. Since $g *_C f = \delta$, it follows by Theorem 5.35 that $f *_C g = \delta$. Thus g is the C-convolution inverse of f, i.e. $g = f^{*_C -1}$. □

Theorem 5.39. *Let $f \in \mathbb{I}[\mathbb{Z}_+, \unlhd]$ be such that*

$$\forall\, x \in \mathbb{Z}_+ : f(x,x) = 1.$$

The C-convolution inverse $f^{*_C -1}$ of the function f is defined recursively as follows:

$$f^{*_C -1}(x,y) = \begin{cases} 1 & \text{if } x = y,\\ -\displaystyle\sum_{\substack{\mathrm{gcf}(z,w)=x\\ \mathrm{lcm}(z,w)=y\\ z\neq y}} f^{*_C -1}(x,z)f(x,w) & \text{if } x \lhd y,\\ 0 & \text{otherwise.} \end{cases}$$

Proof. Follows by Theorem 5.38. □

Lemma 5.18. *If $f \in \mathbb{I}[\mathbb{Z}_+, \unlhd]$ is such that*

$$\forall\, x \in \mathbb{Z}_+ : f(x,x) \neq 0,$$

then

$$\begin{aligned}
&\forall\, x \in \mathbb{Z}_+ : \forall\, p \in \mathbb{P} : \forall\, m,n \in \mathbb{N} :\\
&\qquad f^{*_C -1}(xp^n, xp^{n+m+1}) = -f(x,x)^{-1} f^{*_C -1}(xp^n, xp^n) f(xp^n, xp^{n+m+1}).
\end{aligned}$$

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be such that

$$\forall\, x \in \mathbb{Z}_+ : f(x,x) \neq 0.$$

Let $x \in \mathbb{Z}_+$, $p \in \mathbb{P}$, and $m, n \in \mathbb{N}$. Then $xp^n \lhd xp^{n+m+1}$. Since $\mathrm{gcf}(z,w) = xp^n$ and $\mathrm{lcm}(z,w) = xp^{n+m+1}$ if and only if $z = xp^n$ and $w = xp^{n+m+1}$ or $z = xp^{n+m+1}$ and $w = xp^n$, it follows by Theorem 5.38 that

$$\begin{aligned} f^{*_C-1}(xp^n, xp^{n+m+1}) &= -f(x,x)^{-1} \sum_{\substack{\mathrm{gcf}(z,w)=xp^n \\ \mathrm{lcm}(z,w)=xp^{n+m+1} \\ z\neq xp^{n+m+1}}} f^{*_C-1}(xp^n, z) f(xp^n, w) \\ &= -f(x,x)^{-1} f^{*_C-1}(xp^n, xp^n) f(xp^n, xp^{n+m+1}). \end{aligned}$$

□

Theorem 5.40. *If $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ are invertible with respect to the C-convolution, then $f *_C g$ is invertible with respect to the C-convolution.*

Proof. Let $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be invertible with respect to the C-convolution, and let $x \in \mathbb{Z}_+$. Then by Theorem 5.38 $f(x,x) \neq 0$ and $g(x,x) \neq 0$. Since $(f *_C g)(x,x) = f(x,x)g(x,x)$, it follows that $(f *_C g)(x,x) \neq 0$. Thus by Theorem 5.38 $f *_C g$ is invertible with respect to the C-convolution. □

The following theorem presents the general inversion formula with respect to the C-convolution for $\mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$.

Theorem 5.41. *If $h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is invertible with respect to the C-convolution, then*

$$\forall\, f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq] : f = g *_C h \Leftrightarrow g = f *_C h^{*_C-1}.$$

Proof. Let $h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be invertible with respect to the C-convolution, and let $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. If $f = g *_C h$, then by Theorems 5.36 and 5.34

$$g = g *_C \delta = g *_C (h *_C h^{*_C-1}) = (g *_C h) *_C h^{*_C-1} = f *_C h^{*_C-1}.$$

If $g = f *_C h^{*_C-1}$, then by Theorems 5.36 and 5.34

$$f = f *_C \delta = f *_C (h^{*_C-1} *_C h) = (f *_C h^{*_C-1}) *_C h = g *_C h.$$

□

Theorem 5.42. *Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. If $F \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is the complementary summatory function of the function f, then*

$$f = F *_C \zeta^{*_C-1}, \quad \text{where } \zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq].$$

Proof. Follows by Theorems 5.33 and 5.41. □

Theorem 5.43. *The C-convolution of arithmetic incidence functions distributes over the addition of arithmetic incidence functions.*

Proof. Let $f, g, h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ and $x, y \in \mathbb{Z}_+$. Then

$$\begin{aligned}
[f *_C (g+h)](x,y) &= \sum_{\substack{\mathrm{gcf}(z,w)=x \\ \mathrm{lcm}(z,w)=y}} f(x,z)(g+h)(x,w) \\
&= \sum_{\substack{\mathrm{gcf}(z,w)=x \\ \mathrm{lcm}(z,w)=y}} f(x,z)(g(x,w)+h(x,w)) \\
&= \sum_{\substack{\mathrm{gcf}(z,w)=x \\ \mathrm{lcm}(z,w)=y}} f(x,z)g(x,w)+f(x,z)h(x,w) \\
&= \sum_{\substack{\mathrm{gcf}(z,w)=x \\ \mathrm{lcm}(z,w)=y}} f(x,z)g(x,w)+\sum_{\substack{\mathrm{gcf}(z,w)=x \\ \mathrm{lcm}(z,w)=y}} f(x,z)h(x,w) \\
&= (f *_C g)(x,y) + (f *_C h)(x,y) \\
&= [(f *_C g) + (f *_C h)](x,y).
\end{aligned}$$

Thus $f *_C (g+h) = (f *_C g) + (f *_C h)$, and therefore the left distributive law holds. By Theorem 5.35 and by the left distributive law

$$(f+g) *_C h = h *_C (f+g) = (h *_C f) + (h *_C g) = (f *_C h) + (g *_C h),$$

and therefore the right distributive law holds. Thus the C-convolution of arithmetic incidence functions distributes over the addition of arithmetic incidence functions. □

As is usual for addition and convolution, the addition of arithmetic incidence functions does not distribute over the C-convolution of arithmetic incidence functions. Consequently, following the usual convention, C-convolution is performed before addition in the absence of parentheses, and therefore the related distributive laws take the following form:

$$f *_C (g+h) = f *_C g + f *_C h \quad \text{and} \quad (f+g) *_C h = f *_C h + g *_C h.$$

Theorem 5.44. *The algebraic structure* $\langle \mathbb{I}[\mathbb{Z}_+, \trianglelefteq], +, *_C \rangle$ *is a commutative ring with unity.*

Proof. Follows by Theorems 5.6, 5.37, and 5.43. □

The multiplication of arithmetic incidence functions does not distribute over the C-convolution of arithmetic incidence functions. However, there are such functions in the set $\mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ for which this property holds. The following theorem presents a characterization of such functions.

Theorem 5.45. *Let* $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. *Then*

$$\forall\, g, h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq] : f(g *_C h) = (fg) *_C (fh)$$

if and only if

$$\begin{aligned}\forall\, x, y, z, w \in \mathbb{Z}_+ : \Big[&\mathrm{gcf}(z, w) = x \textit{ and } \mathrm{lcm}(z, w) = y\Big] \\ &\Rightarrow f(x, y) = f(x, z) f(x, w).\end{aligned}$$

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. Let us first assume that

$$\forall\, g, h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq] : f(g *_C h) = (fg) *_C (fh).$$

Let $x, y, z, w \in \mathbb{Z}_+$ be such that $\mathrm{gcf}(z, w) = x$ and $\mathrm{lcm}(z, w) = y$. Let us define the functions $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ as follows:

$$g : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C} : g(u, v) = \begin{cases} 1 & \text{if } u = x \text{ and } v = z, \\ 0 & \text{otherwise,} \end{cases}$$
$$h : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C} : h(u, v) = \begin{cases} 1 & \text{if } u = x \text{ and } v = w, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned}[f(g *_C h)](x, y) &= f(x, y)(g *_C h)(x, y) \\ &= f(x, y) \sum_{\substack{\mathrm{gcf}(s,t) = x \\ \mathrm{lcm}(s,t) = y}} g(x, s) h(x, t) \\ &= f(x, y)[g(x, z) h(x, w)] \\ &= f(x, y),\end{aligned}$$

and

$$\begin{aligned}[(fg) *_C (fh)](x, y) &= \sum_{\substack{\mathrm{gcf}(s,t) = x \\ \mathrm{lcm}(s,t) = y}} (fg)(x, s)(fh)(x, t) \\ &= \sum_{\substack{\mathrm{gcf}(s,t) = x \\ \mathrm{lcm}(s,t) = y}} f(x, s) g(x, s) f(x, t) h(x, t) \\ &= f(x, z) g(x, z) f(x, w) h(x, w) \\ &= f(x, z) f(x, w).\end{aligned}$$

By the assumption $f(g *_C h) = (fg) *_C (fh)$, and therefore

$$f(x, y) = f(x, z) f(x, w).$$

Let us next assume that

$$\forall\, x,y,z,w \in \mathbb{Z}_+ : \Big[\mathrm{gcf}(z,w) = x \text{ and } \mathrm{lcm}(z,w) = y\Big] \\ \Rightarrow f(x,y) = f(x,z)f(x,w).$$

Let $g,h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, and let $x,y \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$. Then by the assumption

$$\begin{aligned}
[f(g *_C h)](x,y) &= f(x,y)(g *_C h)(x,y) \\
&= f(x,y) \sum_{\substack{\mathrm{gcf}(z,w)=x \\ \mathrm{lcm}(z,w)=y}} g(x,z)h(x,w) \\
&= \sum_{\substack{\mathrm{gcf}(z,w)=x \\ \mathrm{lcm}(z,w)=y}} f(x,y)g(x,z)h(x,w) \\
&= \sum_{\substack{\mathrm{gcf}(z,w)=x \\ \mathrm{lcm}(z,w)=y}} f(x,z)f(x,w)g(x,z)h(x,w) \\
&= \sum_{\substack{\mathrm{gcf}(z,w)=x \\ \mathrm{lcm}(z,w)=y}} (fg)(x,z)(fh)(x,w) \\
&= [(fg) *_C (fh)](x,y).
\end{aligned}$$

Thus $f(g *_C h) = (fg) *_C (fh)$. □

The following lemma, used alongside with Lemma 5.10, proves to be very useful when showing that a specific property of an arithmetic incidence function is closed under the C-convolution and the C-convolution inverses.

Lemma 5.19. *Let $x,y,z,w \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$, $z \trianglelefteq w$, and $\mathrm{gcf}(x,z) = \mathrm{gcf}(y,w)$. If $u,v,s,t \in \mathbb{Z}_+$ are such that $x \trianglelefteq u \trianglelefteq y$, $z \trianglelefteq s \trianglelefteq w$, $x \trianglelefteq v \trianglelefteq y$ and $z \trianglelefteq t \trianglelefteq w$, then*

$$\mathrm{gcf}(\mathrm{lcm}(u,s), \mathrm{lcm}(v,t)) = \mathrm{lcm}(x,z), \\ \mathrm{lcm}(\mathrm{lcm}(u,s), \mathrm{lcm}(v,t)) = \mathrm{lcm}(y,w)$$

if and only if

$$\mathrm{gcf}(u,v) = x, \quad \mathrm{lcm}(u,v) = y, \quad \mathrm{gcf}(s,t) = z, \quad \mathrm{lcm}(s,t) = w.$$

Proof. Let $x,y,z,w \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$, $z \trianglelefteq w$, and $\mathrm{gcf}(x,z) = \mathrm{gcf}(y,w)$, and let $u,v,s,t \in \mathbb{Z}_+$ be such that $x \trianglelefteq u \trianglelefteq y$, $z \trianglelefteq s \trianglelefteq w$, $x \trianglelefteq v \trianglelefteq y$ and $z \trianglelefteq t \trianglelefteq w$. Let us first assume that

$$\mathrm{gcf}(\mathrm{lcm}(u,s), \mathrm{lcm}(v,t)) = \mathrm{lcm}(x,z), \\ \mathrm{lcm}(\mathrm{lcm}(u,s), \mathrm{lcm}(v,t)) = \mathrm{lcm}(y,w).$$

Then $\operatorname{lcm}(u,s)\operatorname{lcm}(v,t) = \operatorname{lcm}(x,z)\operatorname{lcm}(y,w)$, and therefore by Lemma 5.10 $uv = xy$ and $st = zw$. Let us note that by Lemma 5.9 the function

$$\beta_1 : [\,x, y\,] \times [\,z, w\,] \to [\,\operatorname{lcm}(x,z), \operatorname{lcm}(y,w)\,] : \beta_1(u,v) = \operatorname{lcm}(u,v)$$

is one-to-one. Since $\operatorname{lcm}(u,v) \in [\,x,y\,]$ and $\operatorname{lcm}(s,t) \in [\,z,w\,]$, and by Theorem 2.17 and the assumption

$$\operatorname{lcm}(\operatorname{lcm}(u,v), \operatorname{lcm}(s,t)) = \operatorname{lcm}(\operatorname{lcm}(u,s), \operatorname{lcm}(v,t)) = \operatorname{lcm}(y,w),$$

it follows that $\operatorname{lcm}(u,v) = y$ and $\operatorname{lcm}(s,t) = w$. Since $uv = xy$ and $st = zw$, it follows by Lemma 5.16 that $\operatorname{gcf}(u,v) = x$ and $\operatorname{gcf}(s,t) = z$.

Let us next assume that

$$\operatorname{gcf}(u,v) = x, \quad \operatorname{lcm}(u,v) = y, \quad \operatorname{gcf}(s,t) = z, \quad \operatorname{lcm}(s,t) = w.$$

Then $uv = xy$ and $st = zw$, and therefore by Lemma 5.10 $\operatorname{lcm}(u,s)\operatorname{lcm}(v,t) = \operatorname{lcm}(x,z)\operatorname{lcm}(y,w)$. By Theorem 2.17

$$\operatorname{lcm}(\operatorname{lcm}(u,s), \operatorname{lcm}(v,t)) = \operatorname{lcm}(\operatorname{lcm}(u,v), \operatorname{lcm}(s,t)) = \operatorname{lcm}(y,w),$$

and therefore by Lemma 5.16 $\operatorname{gcf}(\operatorname{lcm}(u,s), \operatorname{lcm}(v,t)) = \operatorname{lcm}(x,z)$. □

5.8 The Complementary Möbius Function

By Theorem 5.38 the zeta function $\zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ has an inverse with respect to the C-convolution. This C-convolution inverse of the zeta function, being the unitary analogue of the Möbius function, is referred to as the *complementary Möbius function*.

Definition 5.23. The *complementary Möbius function* $\mu_c \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is defined as follows:

$$\mu_c = \zeta^{*_C -1}.$$

Theorem 5.46. *The delta function $\delta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is the complementary summatory function of the complementary Möbius function $\mu_c \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, i.e.*

$$\sum_{\substack{\operatorname{gcf}(z,w)=x \\ \operatorname{lcm}(z,w)=y}} \mu(x,z) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Follows by Theorem 5.33 from the fact that $\mu_c *_C \zeta = \delta$. □

Theorem 5.47. *The complementary Möbius function $\mu_c \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is determined recursively as follows:*

$$\mu_c(x,y) = \begin{cases} 1 & \text{if } x = y, \\ -1 & \text{if } x \blacktriangleleft y, \\ -\displaystyle\sum_{\substack{\gcd(z,w)=x \\ \operatorname{lcm}(z,w)=y \\ z \neq y}} \mu_c(x,z) & \text{if } x \lhd y, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $x, y \in \mathbb{Z}_+$ be such that $x \blacktriangleleft y$. Then $x \lhd y$ and (by Lemma 5.15)

$$\forall\, z, w \in \mathbb{Z}_+ : (\operatorname{gcf}(z,w) = x,\ \operatorname{lcm}(z,w) = y \text{ and } z \neq y) \Rightarrow z = x,$$

and therefore

$$-\sum_{\substack{\operatorname{gcf}(z,w)=x \\ \operatorname{lcm}(z,w)=y \\ z \neq y}} \mu_c(x,z) = -\mu_c(x,x).$$

Thus by Theorem 5.39

$$\mu_c(x,y) = \begin{cases} 1 & \text{if } x = y, \\ -1 & \text{if } x \blacktriangleleft y, \\ -\displaystyle\sum_{\substack{\operatorname{gcf}(z,w)=x \\ \operatorname{lcm}(z,w)=y \\ z \neq y}} \mu_c(x,z) & \text{if } x \lhd y, \\ 0 & \text{otherwise.} \end{cases}$$

□

Theorem 5.48. *The following holds for the complementary Möbius function $\mu_c \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$:*

$$\forall\, x \in \mathbb{Z}_+ : \forall\, p \in \mathbb{P} : \forall\, m, n \in \mathbb{N} : \mu_c(xp^n, xp^{n+m+1}) = -1.$$

Proof. Follows by Lemma 5.18. □

Theorem 5.49. *The complementary Möbius inversion formula for $\mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is the following:*

$$\forall\, f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq] : f = g *_C \zeta \Leftrightarrow g = f *_C \mu_c.$$

Proof. Follows by Theorem 5.41. □

Chapter 6

Translation Invariant Functions

6.1 Translation Invariance

An arithmetic incidence function is constructed in some orderly fashion if it is partially or completely determined by its values at certain elements of the underlying domain, namely the set $\mathbb{Z}_+ \times \mathbb{Z}_+$. There is a variety of properties that reflect the above mentioned aspect of arithmetic incidence functions, and one of them is the notion of translation invariance which captures regularities in function values and their appearances. In this context, in accordance with the theme in focus, the concept of translation invariance builds upon the divisibility of integers.

First, before delving into the details of translation invariance, it is in order to introduce and define the central concept of translation.

Definition 6.1. Let $x, y \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$, and let $u, v \in \mathbb{Z}_+$. The ordered pair $\langle u, v \rangle$ is a *translation* of the ordered pair $\langle x, y \rangle$ if

$$\exists\, z \in \mathbb{Z}_+ : u = xz \text{ and } v = yz.$$

The notion of translation invariance of an arithmetic incidence function can be understood in a stronger sense and a weaker sense. The stronger notion of translation invariance, to be introduced later, is called by the term 'complete translation invariance'. The weaker notion of translation invariance, to be introduced next, is called by the term 'translation invariance', and it is, in effect, a specific application of the notion of translation invariance of an incidence function (see Definition 4.15).

Remark. The notion of translation invariance of an incidence function is introduced by D. A. Smith [49, p. 356], [52, p. 236] and presented also by P. J. McCarthy [25, p. 321]. However, both presentations of this subject matter are very brief in details.

The role of the first of the following two lemmas is to justify the definition of translation invariance, and the role of the second one is to give insights for further characterization of this property.

Lemma 6.1. *Let $x, y, z \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$. Then $\langle \operatorname{lcm}(x,z), \operatorname{lcm}(y,z)\rangle$ is a translation of $\langle x, y\rangle$ if and only if $\operatorname{gcf}(x,z) = \operatorname{gcf}(y,z)$.*

Proof. Let $x, y, z \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$. Let us first assume that $\langle \operatorname{lcm}(x,z), \operatorname{lcm}(y,z)\rangle$ is a translation of $\langle x, y\rangle$. Then $\operatorname{lcm}(x,z) = xw$ and $\operatorname{lcm}(y,z) = yw$, where $w \in \mathbb{Z}_+$. Thus $xz = \operatorname{gcf}(x,z) \cdot xw$ and $yz = \operatorname{gcf}(y,z) \cdot yw$, and therefore $z = \operatorname{gcf}(x,z) \cdot w$ and $z = \operatorname{gcf}(y,z) \cdot w$. Thus $\operatorname{gcf}(x,z) = \operatorname{gcf}(y,z)$.

Let us next assume that $\operatorname{gcf}(x,z) = \operatorname{gcf}(y,z)$. Then $z = \operatorname{gcf}(x,z) \cdot w$ and $z = \operatorname{gcf}(y,z) \cdot w$, where $w \in \mathbb{Z}_+$. Thus $\operatorname{gcf}(x,z) \operatorname{lcm}(x,z) = x \cdot \operatorname{gcf}(x,z) \cdot w$ and $\operatorname{gcf}(y,z) \operatorname{lcm}(y,z) = y \cdot \operatorname{gcf}(y,z) \cdot w$, and therefore $\operatorname{lcm}(x,z) = xw$ and $\operatorname{lcm}(y,z) = yw$. Thus $\langle \operatorname{lcm}(x,z), \operatorname{lcm}(y,z)\rangle$ is a translation of $\langle x, y\rangle$. □

Lemma 6.2. *Let $x, y, z \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$. Then $\langle x, y\rangle$ is a translation of $\langle \operatorname{gcf}(x,z), \operatorname{gcf}(y,z)\rangle$ if and only if $\operatorname{lcm}(x,z) = \operatorname{lcm}(y,z)$.*

Proof. Let $x, y, z \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$. Let us first assume that $\langle x, y\rangle$ is a translation of $\langle \operatorname{gcf}(x,z), \operatorname{gcf}(y,z)\rangle$. Then $x = \operatorname{gcf}(x,z) \cdot w$ and $y = \operatorname{gcf}(y,z) \cdot w$, where $w \in \mathbb{Z}_+$. Thus $xz = \operatorname{gcf}(x,z) \cdot zw$ and $yz = \operatorname{gcf}(y,z) \cdot zw$, and therefore $\operatorname{lcm}(x,z) = zw$ and $\operatorname{lcm}(y,z) = zw$. Thus $\operatorname{lcm}(x,z) = \operatorname{lcm}(y,z)$.

Let us next assume that $\operatorname{lcm}(x,z) = \operatorname{lcm}(y,z)$. Then $\operatorname{lcm}(x,z) = zw$ and $\operatorname{lcm}(y,z) = zw$, where $w \in \mathbb{Z}_+$. Thus $xz = \operatorname{gcf}(x,z) \cdot zw$ and $yz = \operatorname{gcf}(y,z) \cdot zw$, and therefore $x = \operatorname{gcf}(x,z) \cdot w$ and $y = \operatorname{gcf}(y,w) \cdot w$. Thus $\langle x, y\rangle$ is a translation of $\langle \operatorname{gcf}(x,z), \operatorname{gcf}(y,z)\rangle$. □

Definition 6.2. A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is *translation invariant* if

$$\forall\, x, y, z \in \mathbb{Z}_+ : \Big[x \trianglelefteq y \text{ and } \operatorname{gcf}(x,z) = \operatorname{gcf}(y,z)\Big] \\ \Rightarrow f(\operatorname{lcm}(x,z), \operatorname{lcm}(y,z)) = f(x,y).$$

Theorem 6.1. *The zeta function $\zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is translation invariant.*

Proof. Let $x, y, z \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$ and $\operatorname{gcf}(x,z) = \operatorname{gcf}(y,z)$. Then $\operatorname{lcm}(x,z) \trianglelefteq \operatorname{lcm}(y,z)$, and therefore

$$\zeta(\operatorname{lcm}(x,z), \operatorname{lcm}(y,z)) = 1 = \zeta(x,y).$$

Thus $\zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is translation invariant. □

The following two theorems present prime related properties of a translation invariant function, and therefore, in other words, they give necessary conditions that a function must fulfill in order to be translation invariant.

Theorem 6.2. *If $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is translation invariant, then*

$$\forall\, p \in \mathbb{P} : \forall\, m, n \in \mathbb{N} : \\ \Big[m \le n \Rightarrow \forall\, k \in \mathbb{N} : \Big[x = \prod_{\substack{i=1 \\ p_i \in \mathbb{P} \\ p_i \ne p}}^{k} p_i \Rightarrow f(p^m x, p^n x) = f(p^m, p^n)\Big]\Big].$$

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be translation invariant. Let $p \in \mathbb{P}$, and let $m, n \in \mathbb{N}$ be such that $m \leq n$. Let $k \in \mathbb{N}$, and let $x \in \mathbb{Z}_+$ be such that

$$x = \prod_{\substack{i=1 \\ p_i \in \mathbb{P} \\ p_i \neq p}}^{k} p_i.$$

Then $p^m \trianglelefteq p^n$ and

$$\text{gcf}(p^m, x) = 1 \quad \text{and} \quad \text{gcf}(p^n, x) = 1$$

and

$$\text{lcm}(p^m, x) = p^m x \quad \text{and} \quad \text{lcm}(p^n, x) = p^n x.$$

Since $p^m \trianglelefteq p^n$ and $\text{gcf}(p^m, x) = \text{gcf}(p^n, x)$, it follows by the translation invariance of f that

$$f(p^m x, p^n x) = f(\text{lcm}(p^m, x), \text{lcm}(p^n, x)) = f(p^m, p^n).$$

□

In effect, the following theorem follows from Theorem 6.2.

Theorem 6.3. *If $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is translation invariant, then*

$$\forall\, p, q \in \mathbb{P} : \forall\, m, n, k \in \mathbb{N} : (p \neq q \textit{ and } m \leq n) \Rightarrow f(p^m q^k, p^n q^k) = f(p^m, p^n).$$

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be translation invariant. Let $p, q \in \mathbb{P}$ be such that $p \neq q$, and let $m, n, k \in \mathbb{N}$ be such that $m \leq n$. Since $p \neq q$, it follows by Theorem 6.2 that $f(p^m q^k, p^n q^k) = f(p^m, p^n)$. □

Lemma 6.3. *If $x, y, z, w \in \mathbb{Z}_+$ are such that $x \trianglelefteq y$, $z \trianglelefteq w$, and* $\text{gcf}(x, z) = \text{gcf}(y, w)$*, then*

$$\begin{aligned} \forall\, p \in \mathbb{P} : &\Big[\max\{x(p), z(p)\} = x(p),\ \max\{y(p), w(p)\} = y(p),\ \textit{and } z(p) = w(p)\Big] \\ \textit{or } &\Big[\max\{x(p), z(p)\} = z(p),\ \max\{y(p), w(p)\} = w(p),\ \textit{and } x(p) = y(p)\Big], \end{aligned}$$

where $x = \prod_{p \in \mathbb{P}} p^{x(p)}$, $y = \prod_{p \in \mathbb{P}} p^{y(p)}$, $z = \prod_{p \in \mathbb{P}} p^{z(p)}$, and $w = \prod_{p \in \mathbb{P}} p^{w(p)}$.

Proof. Let $x, y, z, w \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$, $z \trianglelefteq w$, and $\text{gcf}(x, z) = \text{gcf}(y, w)$. Since $\text{gcf}(x, z) = \text{gcf}(y, w)$, it follows that

$$\forall\, p \in \mathbb{P} : \min\{x(p), z(p)\} = \min\{y(p), w(p)\}.$$

where $x = \prod_{p \in \mathbb{P}} p^{x(p)}$, $y = \prod_{p \in \mathbb{P}} p^{y(p)}$, $z = \prod_{p \in \mathbb{P}} p^{z(p)}$, and $w = \prod_{p \in \mathbb{P}} p^{w(p)}$. Let $p \in \mathbb{P}$. Then (i) $\max\{y(p), w(p)\} = y(p)$ or (ii) $\max\{y(p), w(p)\} = w(p)$.

(i) Let $\max\{y(p), w(p)\} = y(p)$. Then $\min\{y(p), w(p)\} = w(p)$, and therefore also $\min\{x(p), z(p)\} = w(p)$. Thus $w(p) \leq x(p)$ and $w(p) \leq z(p)$.

Since $z \trianglelefteq w$, it follows that $z(p) \leq w(p)$, and therefore $z(p) = w(p)$. Since $z(p) \leq w(p)$ and $w(p) \leq x(p)$, it follows that $z(p) \leq x(p)$, and therefore $\max\{x(p), z(p)\} = x(p)$.

(ii) Let $\max\{y(p), w(p)\} = w(p)$. Then $\min\{y(p), w(p)\} = y(p)$, and therefore also $\min\{x(p), z(p)\} = y(p)$. Thus $y(p) \leq x(p)$ and $y(p) \leq z(p)$. Since $x \trianglelefteq y$, it follows that $x(p) \leq y(p)$, and therefore $x(p) = y(p)$. Since $x(p) \leq y(p)$ and $y(p) \leq z(p)$, it follows that $x(p) \leq z(p)$, and therefore $\max\{x(p), z(p)\} = z(p)$. $\square$

The following theorem presents, using primes, a sufficient condition for a function to be translation invariant.

Theorem 6.4. *If $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is such that*

(i) $\forall\, x \in \mathbb{Z}_+ : f(x,x) = 1$,

(ii) $\forall\, x, y \in \mathbb{Z}_+ : x \trianglelefteq y \Rightarrow f(x,y) = \prod_{p\in\mathbb{P}} f(p^{x(p)}, p^{y(p)})$,
where $x = \prod_{p\in\mathbb{P}} p^{x(p)}$ *and* $y = \prod_{p\in\mathbb{P}} p^{y(p)}$,

then it is translation invariant.

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be such that (i) and (ii) hold. Let $x, y, z \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$ and $\operatorname{gcf}(x,z) = \operatorname{gcf}(y,z)$. Let us note that $z \trianglelefteq z$, and therefore by Lemma 6.3

$$\begin{aligned}\forall\, p \in \mathbb{P} :& \Big[\max\{x(p), z(p)\} = x(p) \text{ and } \max\{y(p), z(p)\} = y(p)\Big] \\ &\text{or } \Big[\max\{x(p), z(p)\} = z(p),\ \max\{y(p), z(p)\} = z(p), \text{ and } x(p) = y(p)\Big],\end{aligned}$$

where $x = \prod_{p\in\mathbb{P}} p^{x(p)}$, $y = \prod_{p\in\mathbb{P}} p^{y(p)}$, and $z = \prod_{p\in\mathbb{P}} p^{z(p)}$. Let us specifically note that if $\max\{x(p), z(p)\} = z(p)$, $\max\{y(p), z(p)\} = z(p)$, and $x(p) = y(p)$, then by (i)

$$f(p^{\max\{x(p),z(p)\}}, p^{\max\{y(p),z(p)\}}) = f(p^{z(p)}, p^{z(p)}) = 1 = f(p^{x(p)}, p^{y(p)}).$$

Thus

$$\forall\, p \in \mathbb{P} : f(p^{\max\{x(p),z(p)\}}, p^{\max\{y(p),z(p)\}}) = f(p^{x(p)}, p^{y(p)}),$$

and therefore

$$\forall\, p \in \mathbb{P} : f(p^{\operatorname{lcm}(x,z)(p)}, p^{\operatorname{lcm}(y,z)(p)}) = f(p^{x(p)}, p^{y(p)}).$$

Thus by (ii)

$$\begin{aligned}f(\operatorname{lcm}(x,z), \operatorname{lcm}(y,z)) &= \prod_{p\in\mathbb{P}} f(p^{\operatorname{lcm}(x,z)(p)}, p^{\operatorname{lcm}(y,z)(p)}) \\ &= \prod_{p\in\mathbb{P}} f(p^{x(p)}, p^{y(p)}) \\ &= f(x,y).\end{aligned}$$

Thus f is translation invariant. $\square$

The following theorem presents a characterization of a translation invariant function, and therefore, in other words, it gives a necessary and sufficient condition for a function to be translation invariant.

Theorem 6.5. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is translation invariant if and only if*

$$\forall\, x, y \in \mathbb{Z}_+ : f(\mathrm{gcf}(x, y), x) = f(y, \mathrm{lcm}(x, y)).$$

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. Let us first assume that f is translation invariant. Let $x, y \in \mathbb{Z}_+$. Since $\mathrm{gcf}(x, y) \trianglelefteq x$ and $\mathrm{gcf}(\mathrm{gcf}(x, y), y) = \mathrm{gcf}(x, y)$, it follows by the translation invariance of f that

$$f(\mathrm{gcf}(x, y), x) = f(\mathrm{lcm}(\mathrm{gcf}(x, y), y), \mathrm{lcm}(x, y)) = f(y, \mathrm{lcm}(x, y)).$$

Let us next assume that

$$\forall\, x, y \in \mathbb{Z}_+ : f(\mathrm{gcf}(x, y), x) = f(y, \mathrm{lcm}(x, y)).$$

Let $x, y, z \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$ and $\mathrm{gcf}(x, z) = \mathrm{gcf}(y, z)$. Then by Theorems 2.17 and 2.19

$$\begin{aligned} \mathrm{gcf}(y, \mathrm{lcm}(x, z)) &= \mathrm{gcf}(\mathrm{lcm}(x, z), y) \\ &= \mathrm{lcm}(x, \mathrm{gcf}(z, y)) \\ &= \mathrm{lcm}(x, \mathrm{gcf}(y, z)) \\ &= \mathrm{lcm}(x, \mathrm{gcf}(x, z)) \\ &= x, \end{aligned}$$

and therefore by the assumption and Theorem 2.17

$$\begin{aligned} f(x, y) &= f(\mathrm{gcf}(y, \mathrm{lcm}(x, z)), y) \\ &= f(\mathrm{lcm}(x, z), \mathrm{lcm}(y, \mathrm{lcm}(x, z))) \\ &= f(\mathrm{lcm}(x, z), \mathrm{lcm}(\mathrm{lcm}(y, x), z)) \\ &= f(\mathrm{lcm}(x, z), \mathrm{lcm}(y, z)). \end{aligned}$$

Thus f is translation invariant. □

Remark. Theorem 6.5 is a specific application of a general result presented by D. A. Smith [49, pp. 357–358], accompanied with a remark that the result depends only on the local distributivity of the underlying local lattice. The context of that remark considered, the local distributivity cannot be taken as a necessary condition but as a sufficient condition which comes along by the setting chosen by D. A. Smith. Consequently, the above proof of Theorem 6.5 utilizes only the modularity of the factor lattice $(\mathbb{Z}_+, \trianglelefteq)$, demonstrating that the stronger property of distributivity is not actually needed.

Theorem 6.6. *The delta function $\delta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is translation invariant.*

Proof. Let $x, y \in \mathbb{Z}_+$. If $x \trianglelefteq y$, then $\operatorname{gcf}(x,y) = x$ and $\operatorname{lcm}(x,y) = y$, and therefore

$$\delta(\operatorname{gcf}(x,y), x) = \delta(x,x) = 1 = \delta(y,y) = \delta(y, \operatorname{lcm}(x,y)).$$

If $x \not\trianglelefteq y$, then $\operatorname{gcf}(x,y) \neq x$ and $\operatorname{lcm}(x,y) \neq y$, and therefore

$$\delta(\operatorname{gcf}(x,y), x) = 0 = \delta(y, \operatorname{lcm}(x,y)).$$

Thus

$$\forall\, x, y \in \mathbb{Z}_+ : \delta(\operatorname{gcf}(x,y), x) = \delta(y, \operatorname{lcm}(x,y)),$$

and therefore by Theorem 6.5 $\delta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is translation invariant. □

As Theorem 6.5 suggests, the property of a translation invariance has very much to do with the duality of the concepts of the greatest common factor and the least common multiple. This aspect of the translation invariance comes forward by the following theorem which presents another characterization of a translation invariant function (see also Lemma 6.2).

Theorem 6.7. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is translation invariant if and only if*

$$\begin{aligned}\forall\, x, y, z \in \mathbb{Z}_+ : \Big[x \trianglelefteq y \textit{ and } \operatorname{lcm}(x,z) = \operatorname{lcm}(y,z)\Big]&\\ \Rightarrow f(\operatorname{gcf}(x,z), \operatorname{gcf}(y,z)) = f(x,y).&\end{aligned}$$

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. Let us first assume that f is translation invariant. Let $x, y, z \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$ and $\operatorname{lcm}(x,z) = \operatorname{lcm}(y,z)$. Then by Theorems 2.17 and 2.19

$$\begin{aligned}\operatorname{lcm}(\operatorname{gcf}(y,z), x) &= \operatorname{lcm}(x, \operatorname{gcf}(z,y))\\ &= \operatorname{gcf}(\operatorname{lcm}(x,z), y)\\ &= \operatorname{gcf}(\operatorname{lcm}(y,z), y)\\ &= y,\end{aligned}$$

and therefore by Theorems 6.5 and 2.17

$$\begin{aligned}f(x,y) &= f(x, \operatorname{lcm}(\operatorname{gcf}(y,z), x))\\ &= f(\operatorname{gcf}(\operatorname{gcf}(y,z), x), \operatorname{gcf}(y,z))\\ &= f(\operatorname{gcf}(\operatorname{gcf}(x,y), z), \operatorname{gcf}(y,z))\\ &= f(\operatorname{gcf}(x,z), \operatorname{gcf}(y,z)).\end{aligned}$$

Let us next assume that

$$\begin{aligned}\forall\, x, y, z \in \mathbb{Z}_+ : x \trianglelefteq y \text{ and } \operatorname{lcm}(x,z) = \operatorname{lcm}(y,z)&\\ \Rightarrow f(\operatorname{gcf}(x,z), \operatorname{gcf}(y,z)) = f(x,y).&\end{aligned}$$

Let $x, y \in \mathbb{Z}_+$. Since $y \trianglelefteq \operatorname{lcm}(x,y)$ and $\operatorname{lcm}(y,x) = \operatorname{lcm}(\operatorname{lcm}(x,y),x)$, it follows by Theorem 2.17 and the assumption that

$$f(\operatorname{gcf}(x,y),x) = f(\operatorname{gcf}(y,x),\operatorname{gcf}(\operatorname{lcm}(x,y),x)) = f(y,\operatorname{lcm}(x,y)).$$

Thus by Theorem 6.5 f is translation invariant. $\square$

Theorem 6.8. *If $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ are translation invariant, then $f * g$ is translation invariant.*

Proof. Let $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be translation invariant, and let $x, y, z \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$ and $\operatorname{gcf}(x,z) = \operatorname{gcf}(y,z)$. Let us note that by Lemma 5.9 the function

$$\beta_1 : [\,x,y\,] \times [\,z,z\,] \to [\,\operatorname{lcm}(x,z),\operatorname{lcm}(y,z)\,] : \beta_1(u,z) = \operatorname{lcm}(u,z)$$

is one-to-one and onto. Thus by Lemmas 5.9, 5.10, and 5.8 and the translation invariance of f and g

$$\begin{aligned}
&(f*g)(\operatorname{lcm}(x,z),\operatorname{lcm}(y,z))\\
&\qquad = \sum_{\substack{\operatorname{lcm}(x,z)\trianglelefteq q\trianglelefteq \operatorname{lcm}(y,z)\\ qr=\operatorname{lcm}(x,z)\operatorname{lcm}(y,z)}} f(\operatorname{lcm}(x,z),q)g(\operatorname{lcm}(x,z),r)\\
&\qquad \overset{(1)}{=} \sum_{\substack{x\trianglelefteq u\trianglelefteq y\\ x\trianglelefteq v\trianglelefteq y\\ \operatorname{lcm}(u,z)\operatorname{lcm}(v,z)=\operatorname{lcm}(x,z)\operatorname{lcm}(y,z)}} \star\\
&\qquad\quad \star = f(\operatorname{lcm}(x,z),\operatorname{lcm}(u,z))g(\operatorname{lcm}(x,z),\operatorname{lcm}(v,z))\\
&\qquad \overset{(2)}{=} \sum_{\substack{x\trianglelefteq u\trianglelefteq y\\ x\trianglelefteq v\trianglelefteq y\\ uv=xy}} f(\operatorname{lcm}(x,z),\operatorname{lcm}(u,z))g(\operatorname{lcm}(x,z),\operatorname{lcm}(v,z))\\
&\qquad \overset{(3)}{=} \sum_{\substack{x\trianglelefteq u\trianglelefteq y\\ uv=xy}} f(x,u)g(x,v)\\
&\qquad = (f*g)(x,y),
\end{aligned}$$

where (1) means "by Lemma 5.9",
(2) means "by Lemma 5.10",
(3) means "by Lemma 5.8 and the translation invariance of f and g".

Thus $f * g$ is translation invariant. $\square$

Theorem 6.9. *Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be invertible with respect to the D-convolution. If f is translation invariant, then f^{*-1} is translation invariant.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be invertible with respect to the D-convolution and translation invariant, and let $x, y, z \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$ and $\operatorname{gcf}(x, z) = \operatorname{gcf}(y, z)$. Let us note that by Lemma 5.9 the function

$$\beta_1 : [\, x, y\,] \times [\, z, z\,] \to [\, \operatorname{lcm}(x, z), \operatorname{lcm}(y, z)\,] : \beta_1(u, z) = \operatorname{lcm}(u, z)$$

is one-to-one and onto and use induction on the interval $[\, \operatorname{lcm}(x, z), \operatorname{lcm}(y, z)\,]$. If $\operatorname{lcm}(x, z) = \operatorname{lcm}(y, z)$, then $x = y$, since the function β_1 is one-to-one, and therefore by Theorem 5.20 and the translation invariance of f

$$\begin{aligned} f^{*-1}(\operatorname{lcm}(x, z), \operatorname{lcm}(y, z)) &= f(\operatorname{lcm}(x, z), \operatorname{lcm}(y, z))^{-1} \\ &= f(x, y)^{-1} \\ &= f^{*-1}(x, y). \end{aligned}$$

Let $\operatorname{lcm}(x, z) \lhd \operatorname{lcm}(y, z)$. Let us assume that if $u \in \mathbb{Z}_+$ is such that $\operatorname{lcm}(x, z) \trianglelefteq \operatorname{lcm}(u, z) \lhd \operatorname{lcm}(y, z)$, then

$$f^{*-1}(\operatorname{lcm}(x, z), \operatorname{lcm}(u, z)) = f^{*-1}(x, u).$$

Let us note that since the function β_1 is one-to-one, it follows that $\operatorname{lcm}(u, z) = \operatorname{lcm}(y, z)$ if and only if $u = y$, and correspondingly $\operatorname{lcm}(v, z) = \operatorname{lcm}(x, z)$ if and only if $v = x$. Thus by Theorem 5.20, Lemmas 5.3, 5.9 and 5.10, the translation invariance of f, Lemma 5.8, and the induction hypothesis

$$f^{*-1}(\operatorname{lcm}(x, z), \operatorname{lcm}(y, z))$$

$$\overset{(1)}{=} -f(\operatorname{lcm}(x, z), \operatorname{lcm}(x, z))^{-1}\Big(\sum_{\substack{\operatorname{lcm}(x,z) \trianglelefteq q \lhd \operatorname{lcm}(y,z) \\ qr = \operatorname{lcm}(x,z)\, \operatorname{lcm}(y,z)}} \star \Big)$$

$$\star = f^{*-1}(\operatorname{lcm}(x, z), q) f(\operatorname{lcm}(x, z), r)$$

$$\overset{(2)}{=} -f(\operatorname{lcm}(x, z), \operatorname{lcm}(x, z))^{-1}\Big(\sum_{\substack{x \trianglelefteq u \lhd y \\ x \lhd v \trianglelefteq y \\ \operatorname{lcm}(u,z)\, \operatorname{lcm}(v,z) = \operatorname{lcm}(x,z)\, \operatorname{lcm}(y,z)}} \star\star \Big)$$

$$\star\star = f^{*-1}(\operatorname{lcm}(x, z), \operatorname{lcm}(u, z)) f(\operatorname{lcm}(x, z), \operatorname{lcm}(v, z))$$

$$\overset{(3)}{=} -f(\operatorname{lcm}(x, z), \operatorname{lcm}(x, z))^{-1}\Big(\sum_{\substack{x \trianglelefteq u \lhd y \\ x \lhd v \trianglelefteq y \\ uv = xy}} \star\star \Big)$$

$$\overset{(4)}{=} -f(x,x)^{-1} \sum_{\substack{x \lhd u \lhd y \\ uv=xy}} f^{*-1}(x,u) f(x,v)$$
$$\overset{(1)}{=} f^{*-1}(x,y),$$

where (1) means "by Theorem 5.20",
(2) means "by Lemmas 5.3 and 5.9",
(3) means "by Lemma 5.10",
(4) means "by the translation invariance of f, Lemma 5.8, and the induction hypothesis".

Thus f^{*-1} is translation invariant. □

Theorem 6.10. *The Möbius function $\mu \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is translation invariant.*

Proof. Follows by Theorems 6.1 and 6.9. □

Theorem 6.11. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is translation invariant if and only if its divisibility order summatory function is translation invariant.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, and let $F \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be the divisibility order summatory function of the function f. Let us first assume that f is translation invariant. Then by Theorems 6.1 and 6.8 $f * \zeta$ is translation invariant, and therefore by Theorem 5.15 F is translation invariant.

Let us next assume that F is translation invariant. Then by Theorems 6.10 and 6.8 $F * \mu$ is translation invariant, and therefore by Theorem 5.24 f is translation invariant. □

Theorem 6.12. *If $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ are translation invariant, then $f *_C g$ is translation invariant.*

Proof. Let $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be translation invariant, and let $x, y, z \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$ and $\operatorname{gcf}(x,z) = \operatorname{gcf}(y,z)$. Let us note that by Lemma 5.9 the function

$$\beta_1 : [\,x, y\,] \times [\,z, z\,] \to [\,\operatorname{lcm}(x,z), \operatorname{lcm}(y,z)\,] : \beta_1(u,z) = \operatorname{lcm}(u,z)$$

is one-to-one and onto. Thus by Lemmas 5.15, 5.9, 5.10, 5.19, and 5.8 and the translation invariance of f and g

$$
\begin{aligned}
&(f *_C g)(\operatorname{lcm}(x,z), \operatorname{lcm}(y,z)) \\
&\qquad = \sum_{\substack{\operatorname{gcf}(q,r)=\operatorname{lcm}(x,z) \\ \operatorname{lcm}(q,r)=\operatorname{lcm}(y,z)}} f(\operatorname{lcm}(x,z), q) g(\operatorname{lcm}(x,z), r) \\
&\qquad \overset{(1)}{=} \sum_{\substack{\operatorname{lcm}(x,z) \trianglelefteq q \trianglelefteq \operatorname{lcm}(y,z) \\ qr = \operatorname{lcm}(x,z)\operatorname{lcm}(y,z) \\ \operatorname{gcf}(q,r)=\operatorname{lcm}(x,z) \\ \operatorname{lcm}(q,r)=\operatorname{lcm}(y,z)}} f(\operatorname{lcm}(x,z), q) g(\operatorname{lcm}(x,z), r) \\
&\qquad \overset{(2)}{=} \sum_{\substack{x \trianglelefteq u \trianglelefteq y \\ x \trianglelefteq v \trianglelefteq y \\ \operatorname{lcm}(u,z)\operatorname{lcm}(v,z) = \operatorname{lcm}(x,z)\operatorname{lcm}(y,z) \\ \operatorname{gcf}(\operatorname{lcm}(u,z),\operatorname{lcm}(v,z)) = \operatorname{lcm}(x,z) \\ \operatorname{lcm}(\operatorname{lcm}(u,z),\operatorname{lcm}(v,z)) = \operatorname{lcm}(y,z)}} \star
\end{aligned}
$$

$$
\star = f(\operatorname{lcm}(x,z), \operatorname{lcm}(u,z)) g(\operatorname{lcm}(x,z), \operatorname{lcm}(v,z))
$$

$$
\begin{aligned}
&\qquad \overset{(3)}{=} \sum_{\substack{x \trianglelefteq u \trianglelefteq y \\ x \trianglelefteq v \trianglelefteq y \\ uv = xy \\ \operatorname{gcf}(u,v)=x \\ \operatorname{lcm}(u,v)=y}} f(\operatorname{lcm}(x,z), \operatorname{lcm}(u,z)) g(\operatorname{lcm}(x,z), \operatorname{lcm}(v,z)) \\
&\qquad \overset{(4)}{=} \sum_{\substack{x \trianglelefteq u \trianglelefteq y \\ uv = xy \\ \operatorname{gcf}(u,v)=x \\ \operatorname{lcm}(u,v)=y}} f(x,u) g(x,v) \\
&\qquad \overset{(1)}{=} \sum_{\substack{\operatorname{gcf}(u,v)=x \\ \operatorname{lcm}(u,v)=y}} f(x,u) g(x,v) \\
&\qquad = (f *_C g)(x,y),
\end{aligned}
$$

where (1) means “by Lemma 5.15”,
(2) means “by Lemma 5.9”,
(3) means “by Lemmas 5.10 and 5.19”,
(4) means “by Lemma 5.8 and the translation invariance of f and g”.

Thus $f *_C g$ is translation invariant. $\square$

Theorem 6.13. *Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be invertible with respect to the C-convolution. If f is translation invariant, then f^{*_C-1} is translation invariant.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be invertible with respect to the C-convolution and translation invariant, and let $x, y, z \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$ and $\mathrm{gcf}(x, z) = \mathrm{gcf}(y, z)$. Let us note that by Lemma 5.9 the function

$$\beta_1 : [\,x, y\,] \times [\,z, z\,] \to [\,\mathrm{lcm}(x, z), \mathrm{lcm}(y, z)\,] : \beta_1(u, z) = \mathrm{lcm}(u, z)$$

is one-to-one and onto and use induction on the interval $[\,\mathrm{lcm}(x, z), \mathrm{lcm}(y, z)\,]$. If $\mathrm{lcm}(x, z) = \mathrm{lcm}(y, z)$, then $x = y$, since the function β_1 is one-to-one, and therefore by Theorem 5.38 and the translation invariance of f

$$\begin{aligned} f^{*_C-1}(\mathrm{lcm}(x, z), \mathrm{lcm}(y, z)) &= f(\mathrm{lcm}(x, z), \mathrm{lcm}(y, z))^{-1} \\ &= f(x, y)^{-1} \\ &= f^{*_C-1}(x, y). \end{aligned}$$

Let $\mathrm{lcm}(x, z) \vartriangleleft \mathrm{lcm}(y, z)$. Let us assume that if $u \in \mathbb{Z}_+$ is such that $\mathrm{lcm}(x, z) \trianglelefteq \mathrm{lcm}(u, z) \vartriangleleft \mathrm{lcm}(y, z)$, then

$$f^{*_C-1}(\mathrm{lcm}(x, z), \mathrm{lcm}(u, z)) = f^{*_C-1}(x, u).$$

Let us note that since the function β_1 is one-to-one, it follows that $\mathrm{lcm}(u, z) = \mathrm{lcm}(y, z)$ if and only if $u = y$, and correspondingly $\mathrm{lcm}(v, z) = \mathrm{lcm}(x, z)$ if and only if $v = x$. Thus by Theorem 5.38, Lemmas 5.15, 5.3, 5.9, 5.10, and 5.19, the translation invariance of f, Lemma 5.8, and the induction hypothesis

$$\begin{aligned} &f^{*_C-1}(\mathrm{lcm}(x, z), \mathrm{lcm}(y, z)) \\ &\overset{(1)}{=} -f(\mathrm{lcm}(x, z), \mathrm{lcm}(x, z))^{-1}\Big(\sum_{\substack{\mathrm{gcf}(q,r)=\mathrm{lcm}(x,z) \\ \mathrm{lcm}(q,r)=\mathrm{lcm}(y,z) \\ q\neq\mathrm{lcm}(y,z)}} \star \ \Big) \end{aligned}$$

$$\star = f^{*_C-1}(\mathrm{lcm}(x, z), q) f(\mathrm{lcm}(x, z), r)$$

$$\begin{aligned} &\overset{(2)}{=} -f(\mathrm{lcm}(x, z), \mathrm{lcm}(x, z))^{-1}\Big(\sum_{\substack{\mathrm{lcm}(x,z)\trianglelefteq q\vartriangleleft\mathrm{lcm}(y,z) \\ qr=\mathrm{lcm}(x,z)\,\mathrm{lcm}(y,z) \\ \mathrm{gcf}(q,r)=\mathrm{lcm}(x,z) \\ \mathrm{lcm}(q,r)=\mathrm{lcm}(y,z)}} \star \ \Big) \\ &\overset{(3)}{=} -f(\mathrm{lcm}(x, z), \mathrm{lcm}(x, z))^{-1}\Big(\sum_{\substack{x\trianglelefteq u\vartriangleleft y \\ x\vartriangleleft v\trianglelefteq y \\ \mathrm{lcm}(u,z)\,\mathrm{lcm}(v,z)=\mathrm{lcm}(x,z)\,\mathrm{lcm}(y,z) \\ \mathrm{gcf}(\mathrm{lcm}(u,z),\mathrm{lcm}(v,z))=\mathrm{lcm}(x,z) \\ \mathrm{lcm}(\mathrm{lcm}(u,z),\mathrm{lcm}(v,z))=\mathrm{lcm}(y,z)}} \star\star \ \Big) \end{aligned}$$

$$\star\star = f^{*_C-1}(\mathrm{lcm}(x,z),\mathrm{lcm}(u,z))f(\mathrm{lcm}(x,z),\mathrm{lcm}(v,z))$$

$$\overset{(4)}{=} -f(\mathrm{lcm}(x,z),\mathrm{lcm}(x,z))^{-1}\Big(\sum_{\substack{x\trianglelefteq u\triangleleft y\\ x\triangleleft v\trianglelefteq y\\ uv=xy\\ \mathrm{gcf}(u,v)=x\\ \mathrm{lcm}(u,v)=y}} \star\star \Big)$$

$$\overset{(5)}{=} -f(x,x)^{-1} \sum_{\substack{x\trianglelefteq u\triangleleft y\\ uv=xy\\ \mathrm{gcf}(u,v)=x\\ \mathrm{lcm}(u,v)=y}} f^{*_C-1}(x,u)f(x,v)$$

$$\overset{(2)}{=} -f(x,x)^{-1} \sum_{\substack{\mathrm{gcf}(u,v)=x\\ \mathrm{lcm}(u,v)=y\\ u\neq y}} f^{*_C-1}(x,u)f(x,v)$$

$$\overset{(1)}{=} f^{*_C-1}(x,y),$$

where (1) means "by Theorem 5.38",
(2) means "by Lemma 5.15",
(3) means "by Lemmas 5.3 and 5.9",
(4) means "by Lemmas 5.10 and 5.19",
(5) means "by the translation invariance of f, Lemma 5.8, and the induction hypothesis".

Thus f^{*_C-1} is translation invariant. □

Theorem 6.14. *The complementary Möbius function $\mu_c \in \mathbb{I}[\mathbb{Z}_+,\trianglelefteq]$ is translation invariant.*

Proof. Follows by Theorems 6.1 and 6.13. □

Theorem 6.15. *A function $f \in \mathbb{I}[\mathbb{Z}_+,\trianglelefteq]$ is translation invariant if and only if its complementary summatory function is translation invariant.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+,\trianglelefteq]$, and let $F \in \mathbb{I}[\mathbb{Z}_+,\trianglelefteq]$ be the complementary summatory function of the function f. Let us first assume that f is translation invariant. Then by Theorems 6.1 and 6.12 $f *_C \zeta$ is translation invariant, and therefore by Theorem 5.33 F is translation invariant.

Let us next assume that F is translation invariant. Then by Theorems 6.14 and 6.12 $F *_C \mu_c$ is translation invariant, and therefore by Theorem 5.42 f is translation invariant. □

6.2 Complete Translation Invariance

As noted earlier, in addition to the translation invariance of an incidence function, arithmetic incidence functions can be characterized also with a stronger notion of translation invariance. For reasons which will become obvious later, this stronger notion is called by the term 'complete translation invariance'. Although the term 'complete translation invariance' does not appear in the context of incidence functions, the related notion is incidentally referred by D. A. Smith (see e.g. [47, p. 617]) using phrases such as 'a function whose values depend only on the quotient of the arguments'.

The definition of complete translation invariance, like the definition of translation invariance, is based on the definition of a concept of translation (see Definition 6.1).

Definition 6.3. A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is *completely translation invariant* if

$$\forall\, x, y, z \in \mathbb{Z}_+ : x \trianglelefteq y \Rightarrow f(xz, yz) = f(x, y).$$

Theorem 6.16. *The zeta function $\zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely translation invariant.*

Proof. Let $x, y, z \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$. Then $xz \trianglelefteq yz$, and therefore

$$\zeta(xz, yz) = 1 = \zeta(x, y).$$

Thus $\zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely translation invariant. $\square$

The following theorem establishes that if an arithmetic incidence function is completely translation invariant, then it is necessarily also translation invariant.

Theorem 6.17. *If $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely translation invariant, then it is translation invariant.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be completely translation invariant, and let $x, y \in \mathbb{Z}_+$. Since $\operatorname{gcf}(x, y) \trianglelefteq x$ and $y \trianglelefteq \operatorname{lcm}(x, y)$, it follows by the complete translation invariance of f that

$$\begin{aligned} f(\operatorname{gcf}(x, y), x) &= f(\operatorname{gcf}(x, y) \cdot y, x \cdot y) \\ &= f(y \cdot \operatorname{gcf}(x, y), \operatorname{lcm}(x, y) \operatorname{gcf}(x, y)) \\ &= f(y, \operatorname{lcm}(x, y)). \end{aligned}$$

Thus by Theorem 6.5 f is translation invariant. $\square$

The following theorem presents a characterization of a completely translation invariant function, and therefore, in other words, it gives a necessary and sufficient condition for a function to be completely translation invariant.

Theorem 6.18. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely translation invariant if and only if*

$$\forall\, x, y \in \mathbb{Z}_+ : x \trianglelefteq y \Rightarrow f(x, y) = f(1, y/x).$$

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. Let us first assume that f is completely translation invariant. Let $x, y \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$. Since $1 \trianglelefteq y/x$, it follows by the complete translation invariance of f

$$f(1, y/x) = f(1 \cdot x, y/x \cdot x) = f(x, y).$$

Let us next assume that

$$\forall\, x, y \in \mathbb{Z}_+ : x \trianglelefteq y \Rightarrow f(x, y) = f(1, y/x).$$

Let $x, y, z \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$. Since $xz \trianglelefteq yz$ and $yz/xz = y/x$, it follows by the assumption that

$$f(xz, yz) = f(1, yz/xz) = f(1, y/x) = f(x, y).$$

Thus f is completely translation invariant. □

Theorem 6.19. *The delta function $\delta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely translation invariant.*

Proof. Let $x, y \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$. If $x = y$, then $y/x = 1$, and therefore $\delta(x, y) = 1$ and $\delta(1, y/x) = 1$. If $x \neq y$, then $y/x \neq 1$, and therefore $\delta(x, y) = 0$ and $\delta(1, y/x) = 0$. Thus

$$\forall\, x, y \in \mathbb{Z}_+ : x \trianglelefteq y \Rightarrow \delta(x, y) = \delta(1, y/x),$$

and therefore by Theorem 6.18 $\delta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely translation invariant. □

Remark. A completely translation invariant arithmetic incidence function can be 'generated' only from one arithmetic function of one variable. Correspondingly, an arithmetic function of one variable can be 'extracted' only from one completely translation invariant arithmetic incidence function. If the set of completely translation invariant arithmetic incidence functions is denoted by $\mathbb{I}[\mathbb{Z}_+, \trianglelefteq]_{CTI}$, then this one to one correspondence is mediated by the function F, which is one-to-one (injective) and onto (surjective):

$$F : \mathbb{A} \to \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]_{CTI} : F(f) = g,$$

where

$$\forall\, x, y : g(x, y) = \begin{cases} f(y/x) & \text{if } x \trianglelefteq y, \\ 0 & \text{otherwise.} \end{cases}$$

(See also [25, p. 300].)

The following theorem presents another characterization of a completely translation invariant function.

Theorem 6.20. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely translation invariant if and only if*

$$\forall\, x, y, z, w \in \mathbb{Z}_+ : (x \trianglelefteq z \trianglelefteq y \text{ and } zw = xy) \Rightarrow f(x, w) = f(z, y).$$

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. Let us first assume that f is completely translation invariant. Let $x, y, z, w \in \mathbb{Z}_+$ be such that $x \trianglelefteq z \trianglelefteq y$ and $zw = xy$. Then $z = xu$ and $y = wu$, where $u \in \mathbb{Z}_+$. Thus by the complete translation invariance of f

$$f(x, w) = f(xu, wu) = f(z, y).$$

Let us next assume that

$$\forall\, x, y, z, w \in \mathbb{Z}_+ : (x \trianglelefteq z \trianglelefteq y \text{ and } zw = xy) \Rightarrow f(x, w) = f(z, y).$$

Let $x, y, z \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$. Then $x \trianglelefteq xz \trianglelefteq yz$ and $xz \cdot y = x \cdot yz$, and therefore by the assumption $f(x, y) = f(xz, yz)$. Thus f is completely translation invariant. □

Let us explicate the property of a completely translation invariant function stated by Theorem 6.20 in its full potential. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be completely translation invariant, and let $x, y, z, w \in \mathbb{Z}_+$ be such that $x \trianglelefteq z \trianglelefteq y$ and $zw = xy$. Then by Lemma 5.3 $x \trianglelefteq w \trianglelefteq y$, and therefore by Theorem 6.20, besides $f(x, w) = f(z, y)$, also $f(x, z) = f(w, y)$.

Theorem 6.21. *If $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ are completely translation invariant, then*

$$\forall\, x, y \in \mathbb{Z}_+ : (f * g)(x, y) = \sum_{\substack{x \trianglelefteq z \trianglelefteq y \\ zw = xy}} f(z, y) g(w, y).$$

Proof. Let $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be completely translation invariant, and let $x, y \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$. Then by Lemma 5.5 and Theorem 6.20

$$\begin{aligned}(f * g)(x, y) &= \sum_{\substack{x \trianglelefteq z \trianglelefteq y \\ zw = xy}} f(x, z) g(x, w) \\ &= \sum_{\substack{x \trianglelefteq z \trianglelefteq y \\ zw = xy}} f(x, w) g(x, z) \\ &= \sum_{\substack{x \trianglelefteq z \trianglelefteq y \\ zw = xy}} f(z, y) g(w, y).\end{aligned}$$

□

Theorem 6.22. *Let $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. If g is completely translation invariant, then*

$$\forall\, x, y \in \mathbb{Z}_+ : (f * g)(x, y) = \sum_{x \trianglelefteq z \trianglelefteq y} f(x, z)g(z, y).$$

Proof. Let $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, and let g be completely translation invariant, and let $x, y \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$. Then by Theorem 6.20

$$(f * g)(x, y) = \sum_{\substack{x \trianglelefteq z \trianglelefteq y \\ zw = xy}} f(x, z)g(x, w) = \sum_{x \trianglelefteq z \trianglelefteq y} f(x, z)g(z, y).$$

□

Remark. Theorem 6.22 establishes that in the set of completely translation invariant arithmetic incidence functions the D-convolution and the convolution of incidence functions are essentially one and the same operation.

Theorem 6.23. *If $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ are completely translation invariant, then $f * g$ is completely translation invariant.*

Proof. Let $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be completely translation invariant, and let $x, y, z \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$. Let us note that if $q \in \mathbb{Z}_+$ is such that $xz \trianglelefteq q \trianglelefteq yz$, then $q = uz$, where $u \in \mathbb{Z}_+$. Let us also note that $x \trianglelefteq u \trianglelefteq y$ if and only if $xz \trianglelefteq uz \trianglelefteq yz$. Thus by the complete translation invariance of f and g

$$\begin{aligned}(f * g)(xz, yz) &= \sum_{\substack{xz \trianglelefteq q \trianglelefteq yz \\ qr = (xz)(yz)}} f(xz, q)g(xz, r) \\ &= \sum_{\substack{xz \trianglelefteq uz \trianglelefteq yz \\ (uz)(vz) = (xz)(yz)}} f(xz, uz)g(xz, vz) \\ &= \sum_{\substack{x \trianglelefteq u \trianglelefteq y \\ uv = xy}} f(x, u)g(x, v) \\ &= (f * g)(x, y).\end{aligned}$$

Thus $f * g$ is completely translation invariant. □

Theorem 6.24. *Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be invertible with respect to the D-convolution. If f is completely translation invariant, then f^{*-1} is completely translation invariant.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be invertible with respect to the D-convolution and translation invariant, and let $x, y, z \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$. Let us use induction on the interval $[xz, yz]$. If $xz = yz$, then $x = y$, and therefore by Theorem 5.20 and the complete translation invariance of f

$$f^{*-1}(xz, yz) = f(xz, yz)^{-1} = f(x, y)^{-1} = f^{*-1}(x, y).$$

Let $xz \lhd yz$. Let us assume that if $u \in \mathbb{Z}_+$ is such that $xz \unlhd uz \lhd yz$, then

$$f^{*-1}(xz, uz) = f^{*-1}(x, u).$$

Let us note that if $q \in \mathbb{Z}_+$ is such that $xz \unlhd q \unlhd yz$, then $q = uz$, where $u \in \mathbb{Z}_+$. Let us also note that $x \unlhd u \unlhd y$ if and only if $xz \unlhd uz \unlhd yz$. Thus by Theorem 5.20, the complete translation invariance of f, and the induction hypothesis,

$$\begin{aligned} f^{*-1}(xz, yz) &= -f(xz, xz)^{-1} \sum_{\substack{xz \unlhd q \lhd yz \\ qr=(xz)(yz)}} f^{*-1}(xz, q) f(xz, r) \\ &= -f(xz, xz)^{-1} \sum_{\substack{xz \unlhd uz \lhd yz \\ (uz)(vz)=(xz)(yz)}} f^{*-1}(xz, uz) f(xz, vz) \\ &= -f(x, x)^{-1} \sum_{\substack{x \unlhd u \lhd y \\ uv=xy}} f^{*-1}(x, u) f(x, v) \\ &= f^{*-1}(x, y). \end{aligned}$$

Thus f^{*-1} is completely translation invariant. □

Theorem 6.25. *The Möbius function $\mu \in \mathbb{I}[\mathbb{Z}_+, \unlhd]$ is completely translation invariant.*

Proof. Follows by Theorems 6.16 and 6.24. □

Theorem 6.26. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \unlhd]$ is completely translation invariant if and only if its divisibility order summatory function is completely translation invariant.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \unlhd]$, and let $F \in \mathbb{I}[\mathbb{Z}_+, \unlhd]$ be the divisibility order summatory function of the function f. Let us first assume that f is completely translation invariant. Then by Theorems 6.16 and 6.23 $f * \zeta$ is completely translation invariant, and therefore by Theorem 5.15 F is completely translation invariant.

Let us next assume that F is completely translation invariant. Then by Theorems 6.25 and 6.23 $F * \mu$ is completely translation invariant, and therefore by Theorem 5.24 f is completely translation invariant. □

Theorem 6.27. *If $f, g \in \mathbb{I}[\mathbb{Z}_+, \unlhd]$ are completely translation invariant, then $f *_C g$ is completely translation invariant.*

Proof. Let $f, g \in \mathbb{I}[\mathbb{Z}_+, \unlhd]$ be completely translation invariant, and let $x, y, z \in \mathbb{Z}_+$ be such that $x \unlhd y$. Let us note that if $q \in \mathbb{Z}_+$ is such that $xz \unlhd q \unlhd yz$, then $q = uz$, where $u \in \mathbb{Z}_+$. Let us also note that $x \unlhd u \unlhd y$ if and only if

$xz \trianglelefteq uz \trianglelefteq yz$. Thus by Lemma 5.15 and the complete translation invariance of f and g

$$\begin{aligned}
(f *_C g)(xz, yz) &= \sum_{\substack{\gcd(q,r)=xz \\ \operatorname{lcm}(q,r)=yz}} f(xz, q)g(xz, r) \\
&= \sum_{\substack{xz \trianglelefteq q \trianglelefteq yz \\ qr=(xz)(yz) \\ \gcd(q,r)=xz \\ \operatorname{lcm}(q,r)=yz}} f(xz, q)g(xz, r) \\
&= \sum_{\substack{xz \trianglelefteq uz \trianglelefteq yz \\ (uz)(vz)=(xz)(yz) \\ \gcd(uz,vz)=xz \\ \operatorname{lcm}(uz,vz)=yz}} f(xz, uz)g(xz, vz) \\
&= \sum_{\substack{x \trianglelefteq u \trianglelefteq y \\ uv=xy \\ \gcd(u,v)=x \\ \operatorname{lcm}(u,v)=y}} f(x, u)g(x, v) \\
&= \sum_{\substack{\gcd(u,v)=x \\ \operatorname{lcm}(u,v)=y}} f(x, u)g(x, v) \\
&= (f *_C g)(x, y).
\end{aligned}$$

Thus $f *_C g$ is completely translation invariant. □

Theorem 6.28. *Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be invertible with respect to the C-convolution. If f is completely translation invariant, then $f^{*_C -1}$ is completely translation invariant.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be invertible with respect to the C-convolution and translation invariant, and let $x, y, z \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$. Let us use induction on the interval $[xz, yz]$. If $xz = yz$, then $x = y$, and therefore by Theorem 5.38 and the complete translation invariance of f

$$f^{*_C -1}(xz, yz) = f(xz, yz)^{-1} = f(x, y)^{-1} = f^{*_C -1}(x, y).$$

Let $xz \triangleleft yz$. Let us assume that if $u \in \mathbb{Z}_+$ is such that $xz \trianglelefteq uz \triangleleft yz$, then

$$f^{*_C -1}(xz, uz) = f^{*_C -1}(x, u).$$

Let us note that if $q \in \mathbb{Z}_+$ is such that $xz \trianglelefteq q \trianglelefteq yz$, then $q = uz$, where $u \in \mathbb{Z}_+$. Let us also note that $x \trianglelefteq u \trianglelefteq y$ if and only if $xz \trianglelefteq uz \trianglelefteq yz$. Thus by Theorem 5.38, Lemma 5.15, the complete translation invariance of f, and the induction hypothesis,

$$\begin{aligned}
f^{*_C-1}(xz,yz) &= -f(xz,xz)^{-1} \sum_{\substack{\operatorname{gcf}(q,r)=xz \\ \operatorname{lcm}(q,r)=yz \\ q\neq yz}} f^{*_C-1}(xz,q)f(xz,r) \\
&= -f(xz,xz)^{-1} \sum_{\substack{xz\trianglelefteq q\triangleleft yz \\ qr=(xz)(yz) \\ \operatorname{gcf}(q,r)=xz \\ \operatorname{lcm}(q,r)=yz}} f^{*_C-1}(xz,q)f(xz,r) \\
&= -f(xz,xz)^{-1} \sum_{\substack{xz\trianglelefteq uz\triangleleft yz \\ (uz)(vz)=(xz)(yz) \\ \operatorname{gcf}(uz,vz)=xz \\ \operatorname{lcm}(uz,vz)=yz}} f^{*_C-1}(xz,uz)f(xz,vz) \\
&= -f(x,x)^{-1} \sum_{\substack{x\trianglelefteq u\triangleleft y \\ uv=xy \\ \operatorname{gcf}(u,v)=x \\ \operatorname{lcm}(u,v)=y}} f^{*_C-1}(x,u)f(x,v) \\
&= -f(x,x)^{-1} \sum_{\substack{\operatorname{gcf}(u,v)=x \\ \operatorname{lcm}(u,v)=y \\ u\neq y}} f^{*_C-1}(x,u)f(x,v) \\
&= f^{*_C-1}(x,y).
\end{aligned}$$

Thus f^{*_C-1} is completely translation invariant. □

Theorem 6.29. *The complementary Möbius function $\mu_c \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely translation invariant.*

Proof. Follows by Theorems 6.16 and 6.28. □

Theorem 6.30. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely translation invariant if and only if its complementary summatory function is completely translation invariant.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, and let $F \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be the complementary summatory function of the function f. Let us first assume that f is completely translation invariant. Then by Theorems 6.16 and 6.27 $f *_C \zeta$ is completely translation invariant, and therefore by Theorem 5.33 F is completely translation invariant.

Let us next assume that F is completely translation invariant. Then by Theorems 6.29 and 6.27 $F *_C \mu_c$ is completely translation invariant, and therefore by Theorem 5.42 f is completely translation invariant. □

Chapter 7

Semifactorable Functions

7.1 Semifactorability

The notion of semifactorability of an arithmetic incidence function generalizes the notion of multiplicativity of an arithmetic function of one variable. This generalization builds upon the underlying partial ordering $\trianglelefteq$ and the related lattice operations gcf and lcm.

Definition 7.1. A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is *semifactorable* if

(i) $\forall\, x \in \mathbb{Z}_+ : f(x, x) = 1$,

(ii) $\forall\, x, y, w \in \mathbb{Z}_+ : x = \mathrm{gcf}(y, w) \Rightarrow f(x, \mathrm{lcm}(y, w)) = f(x, y)f(x, w)$.

Remark. The use of the term 'factorable' is motivated by the way how the function values at specific arguments satisfying the required conditions can be broken down, in other words factored, into a product of function values at arguments that are, in effect, factors of the original arguments.

Remark. The notion of semifactorability of an incidence function is presented by D. A. Smith [49, p. 356], [52, p. 236] and by P. J. McCarthy [25, p. 321], although they do not call it by any specific term. However, both presentations of this subject matter are very brief in details compared to the present study.

Theorem 7.1. *The zeta function $\zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is semifactorable.*

Proof. (i) Let $x \in \mathbb{Z}_+$. Since $x \trianglelefteq x$, it follows that $\zeta(x, x) = 1$. (ii) Let $x, y, w \in \mathbb{Z}_+$ be such that $x = \mathrm{gcf}(y, w)$. Then $x \trianglelefteq \mathrm{lcm}(y, w)$, $x \trianglelefteq y$, and $x \trianglelefteq w$, and therefore

$$\zeta(x, \mathrm{lcm}(y, w)) = 1 = \zeta(x, y)\zeta(x, w).$$

By (i) and (ii) $\zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is semifactorable. $\square$

The following theorem presents a characterization of a semifactorable function, and it is essentially a reformulation of the definition of a semifactorable function.

Theorem 7.2. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is semifactorable if and only if*

(i) $\forall\, x \in \mathbb{Z}_+ : f(x,x) = 1$,

(ii) $\forall\, x, y, z, w \in \mathbb{Z}_+ : \Big[\mathrm{gcf}(z,w) = x \text{ and } \mathrm{lcm}(z,w) = y\Big]$
$\Rightarrow f(x,y) = f(x,z)f(x,w)$.

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. Let us first assume that f is semifactorable. (i) By the semifactorability of f

$$\forall\, x \in \mathbb{Z}_+ : f(x,x) = 1.$$

(ii) Let $x, y, z, w \in \mathbb{Z}_+$ be such that $\mathrm{gcf}(z,w) = x$ and $\mathrm{lcm}(z,w) = y$. Then by the semifactorability of f

$$f(x,y) = f(x, \mathrm{lcm}(z,w)) = f(x,z)f(x,w).$$

Thus (i) and (ii) hold.

Let us next assume that (i) and (ii) hold. Let $x, y, w \in \mathbb{Z}_+$ be such that $x = \mathrm{gcf}(y,w)$. Then by (ii) $f(x, \mathrm{lcm}(y,w)) = f(x,y)f(x,w)$. Thus f is semifactorable. □

Theorem 7.3. *The delta function $\delta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is semifactorable.*

Proof. Let $x, y, z, w \in \mathbb{Z}_+$ be such that $\mathrm{gcf}(z,w) = x$ and $\mathrm{lcm}(z,w) = y$. Then by Lemmas 5.15 and 5.3 $x \trianglelefteq z \trianglelefteq y$, $x \trianglelefteq w \trianglelefteq y$, and $zw = xy$. Let us assume that $x = y$. Since $x \trianglelefteq z \trianglelefteq y$ and $x \trianglelefteq w \trianglelefteq y$, it follows that $x = z$ and $x = w$, and therefore

$$\delta(x,y) = 1 = \delta(x,z)\delta(x,w).$$

Let us assume that $x \neq y$. Since $zw = xy$, it follows that $x = z$ if and only if $y = w$. Thus $\delta(x,z) = 1$ if and only if $\delta(x,w) = 0$, and therefore

$$\delta(x,y) = 0 = \delta(x,z)\delta(x,w).$$

Thus by Theorem 7.2 $\delta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is semifactorable. □

Lemma 7.1. *If $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is such that*

$$\forall\, x, y \in \mathbb{Z}_+ : x \trianglelefteq y \Rightarrow f(x,y) = \prod_{p \in \mathbb{P}} f(p^{x(p)}, p^{y(p)}),$$

where $x = \prod_{p\in\mathbb{P}} p^{x(p)}$ and $y = \prod_{p\in\mathbb{P}} p^{y(p)}$, then

$$\forall\, x, y, z, w \in \mathbb{Z}_+ : \Big[x \trianglelefteq y,\ z \trianglelefteq w \text{ and } \mathrm{gcf}(x,z) = \mathrm{gcf}(y,w)\Big]$$
$$\Rightarrow f(\mathrm{gcf}(x,z), \mathrm{gcf}(y,w)) f(\mathrm{lcm}(x,z), \mathrm{lcm}(y,w)) = f(x,y)f(z,w).$$

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be such that

$$\forall\, x, y \in \mathbb{Z}_+ : x \trianglelefteq y \Rightarrow f(x, y) = \prod_{p\in\mathbb{P}} f(p^{x(p)}, p^{y(p)}),$$

where $x = \prod_{p\in\mathbb{P}} p^{x(p)}$ and $y = \prod_{p\in\mathbb{P}} p^{y(p)}$. Let $x, y, z, w \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$, $z \trianglelefteq w$, and $\operatorname{gcf}(x, z) = \operatorname{gcf}(y, w)$. Let us note that by Lemma 6.3

$$\begin{aligned}\forall\, p \in \mathbb{P} :& \Big[\max\{x(p), z(p)\} = x(p) \text{ and } \max\{y(p), w(p)\} = y(p)\Big] \\ &\text{or } \Big[\max\{x(p), z(p)\} = z(p) \text{ and } \max\{y(p), w(p)\} = w(p)\Big],\end{aligned}$$

where $x = \prod_{p\in\mathbb{P}} p^{x(p)}$, $y = \prod_{p\in\mathbb{P}} p^{y(p)}$, $z = \prod_{p\in\mathbb{P}} p^{z(p)}$, and $w = \prod_{p\in\mathbb{P}} p^{w(p)}$. Let us specifically note that if $\max\{x(p), z(p)\} = x(p)$ and $\max\{y(p), w(p)\} = y(p)$, then $\min\{x(p), z(p)\} = z(p)$ and $\min\{y(p), w(p)\} = w(p)$, and therefore

$$f(p^{\max\{x(p),z(p)\}}, p^{\max\{y(p),w(p)\}}) = f(p^{x(p)}, p^{y(p)})$$

and

$$f(p^{\min\{x(p),z(p)\}}, p^{\min\{y(p),w(p)\}}) = f(p^{z(p)}, p^{w(p)}).$$

Correspondingly, if $\max\{x(p), z(p)\} = z(p)$ and $\max\{y(p), w(p)\} = w(p)$, then $\min\{x(p), z(p)\} = x(p)$ and $\min\{y(p), w(p)\} = y(p)$, and therefore

$$f(p^{\max\{x(p),z(p)\}}, p^{\max\{y(p),w(p)\}}) = f(p^{z(p)}, p^{w(p)})$$

and

$$f(p^{\min\{x(p),z(p)\}}, p^{\min\{y(p),w(p)\}}) = f(p^{x(p)}, p^{y(p)}).$$

Thus

$$\begin{aligned}\forall\, p \in \mathbb{P} :\ & f(p^{\min\{x(p),z(p)\}}, p^{\min\{y(p),w(p)\}}) f(p^{\max\{x(p),z(p)\}}, p^{\max\{y(p),w(p)\}}) \\ &= f(p^{x(p)}, p^{y(p)}) f(p^{z(p)}, p^{w(p)}),\end{aligned}$$

and therefore

$$\begin{aligned}\forall\, p \in \mathbb{P} :\ & f(p^{\operatorname{gcf}(x,z)(p)}, p^{\operatorname{gcf}(y,w)(p)}) f(p^{\operatorname{lcm}(x,z)(p)}, p^{\operatorname{lcm}(y,w)(p)}) \\ &= f(p^{x(p)}, p^{y(p)}) f(p^{z(p)}, p^{w(p)}).\end{aligned}$$

Thus by the assumption

$$\begin{aligned}& f(\operatorname{gcf}(x, z), \operatorname{gcf}(y, w)) f(\operatorname{lcm}(x, z), \operatorname{lcm}(y, w)) \\ &\quad = \Big[\prod_{p\in\mathbb{P}} f(p^{\operatorname{gcf}(x,z)(p)}, p^{\operatorname{gcf}(y,w)(p)})\Big]\Big[\prod_{p\in\mathbb{P}} f(p^{\operatorname{lcm}(x,z)(p)}, p^{\operatorname{lcm}(y,w)(p)})\Big] \\ &\quad = \prod_{p\in\mathbb{P}} f(p^{\operatorname{gcf}(x,z)(p)}, p^{\operatorname{gcf}(y,w)(p)}) f(p^{\operatorname{lcm}(x,z)(p)}, p^{\operatorname{lcm}(y,w)(p)}) \\ &\quad = \prod_{p\in\mathbb{P}} f(p^{x(p)}, p^{y(p)}) f(p^{z(p)}, p^{w(p)}) \\ &\quad = \Big[\prod_{p\in\mathbb{P}} f(p^{x(p)}, p^{y(p)})\Big]\Big[\prod_{p\in\mathbb{P}} f(p^{z(p)}, p^{w(p)})\Big] \\ &\quad = f(x, y) f(z, w).\end{aligned}$$

□

The following theorem presents, using primes, a sufficient condition for a function to be semifactorable (see Theorem 6.4).

Theorem 7.4. *If $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is such that*

(i) $\forall\, x \in \mathbb{Z}_+ : f(x, x) = 1$,

(ii) $\forall\, x, y \in \mathbb{Z}_+ : x \trianglelefteq y \Rightarrow f(x, y) = \prod_{p\in\mathbb{P}} f(p^{x(p)}, p^{y(p)})$,
where $x = \prod_{p\in\mathbb{P}} p^{x(p)}$ *and* $y = \prod_{p\in\mathbb{P}} p^{y(p)}$,

then it is semifactorable.

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be such that (i) and (ii) hold. Let $x, y, w \in \mathbb{Z}_+$ be such that $x = \mathrm{gcf}(y, w)$. Then $x \trianglelefteq y$, $x \trianglelefteq w$, and $\mathrm{gcf}(x, x) = \mathrm{gcf}(y, w)$. Thus by (ii) and Lemma 7.1

$$f(\mathrm{gcf}(x, x), \mathrm{gcf}(y, w))f(\mathrm{lcm}(x, x), \mathrm{lcm}(y, w)) = f(x, y)f(x, w),$$

and therefore by (i) $f(x, \mathrm{lcm}(y, w)) = f(x, y)f(x, w)$. Thus f is semifactorable. □

Lemma 7.2. *Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. If $x, y, w \in \mathbb{Z}_+$ are such that $x = \mathrm{gcf}(y, w)$ and $f(x, x) = 1$, then*

$$\forall\, p \in \mathbb{P} : f(x, xp^{\mathrm{lcm}(y,w)(p)-x(p)}) = f(x, xp^{y(p)-x(p)})f(x, xp^{w(p)-x(p)}).$$

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, and let $x, y, w \in \mathbb{Z}_+$ be such that $x = \mathrm{gcf}(y, w)$ and $f(x, x) = 1$. Let us note that since $x \trianglelefteq y$, $x \trianglelefteq w$, and $\mathrm{gcf}(x, x) = \mathrm{gcf}(y, w)$, it follows by Lemma 6.3 that

$$\begin{aligned} \forall\, p \in \mathbb{P} :& \Big[\max\{y(p), w(p)\} = y(p) \text{ and } x(p) = w(p)\Big] \\ &\text{or } \Big[\max\{y(p), w(p)\} = w(p) \text{ and } x(p) = y(p)\Big], \end{aligned}$$

where $x = \prod_{p\in\mathbb{P}} p^{x(p)}$, $y = \prod_{p\in\mathbb{P}} p^{y(p)}$, and $w = \prod_{p\in\mathbb{P}} p^{w(p)}$. Let us specifically note that if $\max\{y(p), w(p)\} = y(p)$ and $x(p) = w(p)$, then

$$\begin{aligned} f(x, xp^{\max\{y(p),w(p)\}-x(p)}) &= f(x, xp^{y(p)-x(p)})f(x, x) \\ &= f(x, xp^{y(p)-x(p)})f(x, xp^{w(p)-x(p)}). \end{aligned}$$

Correspondingly, if $\max\{y(p), w(p)\} = w(p)$ and $x(p) = y(p)$, then

$$\begin{aligned} f(x, xp^{\max\{y(p),w(p)\}-x(p)}) &= f(x, x)f(x, xp^{w(p)-x(p)}) \\ &= f(x, xp^{y(p)-x(p)})f(x, xp^{w(p)-x(p)}). \end{aligned}$$

Thus

$$\forall\, p \in \mathbb{P} : f(x, xp^{\max\{y(p),w(p)\}-x(p)}) = f(x, xp^{y(p)-x(p)})f(x, xp^{w(p)-x(p)}),$$

and therefore

$$\forall\, p \in \mathbb{P} : f(x, xp^{\mathrm{lcm}(y,w)(p)-x(p)}) = f(x, xp^{y(p)-x(p)})f(x, xp^{w(p)-x(p)}).$$

□

The following theorem presents a prime related characterization of a semifactorable function, and therefore, in other words, it gives a necessary and sufficient condition for a function to be semifactorable.

Theorem 7.5. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is semifactorable if and only if*

(i) $\forall\, x \in \mathbb{Z}_+ : f(x,x) = 1$,

(ii) $\forall\, x, y \in \mathbb{Z}_+ : x \trianglelefteq y \Rightarrow f(x,y) = \prod_{p\in\mathbb{P}} f(x, xp^{y(p)-x(p)})$,
where $x = \prod_{p\in\mathbb{P}} p^{x(p)}$ *and* $y = \prod_{p\in\mathbb{P}} p^{y(p)}$.

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. Let us first assume that f is semifactorable. (i) By the semifactorability of f

$$\forall\, x \in \mathbb{Z}_+ : f(x,x) = 1.$$

(ii) Let $x, y \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$, where $x = \prod_{p\in\mathbb{P}} p^{x(p)}$ and $y = \prod_{p\in\mathbb{P}} p^{y(p)}$. Let us assume that $y = 1$. Then $x = 1$. Thus $x(p) = 0$ and $y(p) = 0$ for all $p \in \mathbb{P}$, and therefore by (i)

$$f(x,y) = f(1,1) = \prod_{p\in\mathbb{P}} f(1,1) = \prod_{p\in\mathbb{P}} f(x, xp^{y(p)-x(p)}).$$

Let us assume that $y \neq 1$. Then

$$x = \prod_{p\in\mathbb{P}} p^{x(p)} = \prod_{i=1}^{n} p_i^{x(p_i)} \quad \text{and} \quad y = \prod_{p\in\mathbb{P}} p^{y(p)} = \prod_{i=1}^{n} p_i^{y(p_i)},$$

where $n \in \mathbb{Z}_+$ stands for the number of distinct primes in the prime factorization of the element y and $x(p_i) = 0$, $1 \leq i \leq n$, if $p_i \in \mathbb{P}$ is not a prime factor of the element x. Let us note that

$$\forall\, p \in \mathbb{P} : 0 \leq x(p) \leq y(p),$$

and therefore for all $n \in \mathbb{Z}_+$

$$y = x \prod_{i=1}^{n} p_i^{y(p_i)-x(p_i)}.$$

Let us use induction on $n \in \mathbb{Z}_+$ to show that

$$\forall\, n \in \mathbb{Z}_+ : f(x,y) = \prod_{i=1}^{n} f(x, xp_i^{y(p_i)-x(p_i)})$$

from which the result follows. If $n = 1$, then

$$f(x,y) = f(x, x\prod_{i=1}^{1} p_i^{y(p_i)-x(p_i)}) = f(x, xp^{y(p_1)-x(p_1)}) = \prod_{i=1}^{1} f(x, xp_i^{y(p_i)-x(p_i)}).$$

Let $n \in \mathbb{Z}_+$, and let us assume that the claim holds for n, that is,

$$f(x, x\prod_{i=1}^{n} p_i^{y(p_i)-x(p_i)}) = \prod_{i=1}^{n} f(x, xp_i^{y(p_i)-x(p_i)}).$$

Let the number of distinct primes in the prime factorization of the element y be $n+1$. Then

$$x = \prod_{i=1}^{n+1} p_i^{x(p_i)} \quad \text{and} \quad y = \prod_{i=1}^{n+1} p_i^{y(p_i)}.$$

Since

$$\operatorname{gcf}(x\prod_{i=1}^{n} p_i^{y(p_i)-x(p_i)}, xp_{n+1}^{y(p_{n+1})-x(p_{n+1})}) = x$$

and

$$\operatorname{lcm}(x\prod_{i=1}^{n} p_i^{y(p_i)-x(p_i)}, xp_{n+1}^{y(p_{n+1})-x(p_{n+1})}) = x\prod_{i=1}^{n+1} p_i^{y(p_i)-x(p_i)} = y,$$

it follows by the semifactorability of f and the induction hypothesis that

$$\begin{aligned}
f(x,y) &= f(x, x\prod_{i=1}^{n+1} p_i^{y(p_i)-x(p_i)}) \\
&= f(x, \operatorname{lcm}(x\prod_{i=1}^{n} p_i^{y(p_i)-x(p_i)}, xp_{n+1}^{y(p_{n+1})-x(p_{n+1})})) \\
&= f(x, x\prod_{i=1}^{n} p_i^{y(p_i)-x(p_i)}) f(x, xp_{n+1}^{y(p_{n+1})-x(p_{n+1})}) \\
&= \Big[\prod_{i=1}^{n} f(x, xp_i^{y(p_i)-x(p_i)})\Big] f(x, xp_{n+1}^{y(p_{n+1})-x(p_{n+1})}) \\
&= \prod_{i=1}^{n+1} f(x, xp_i^{y(p_i)-x(p_i)}).
\end{aligned}$$

Thus

$$\forall\, n \in \mathbb{Z}_+ : f(x,y) = \prod_{i=1}^{n} f(x, xp_i^{y(p_i)-x(p_i)}).$$

From the semifactorability of f it follows that

$$\forall\, p \in \mathbb{P} : x(p) = y(p) = 0 \Rightarrow f(x, xp^{y(p)-x(p)}) = 1,$$

and therefore

$$f(x,y) = \prod_{p \in \mathbb{P}} f(x, xp^{y(p)-x(p)}).$$

Thus (i) and (ii) hold.

Let us next assume that (i) and (ii) hold. Let $x, y, w \in \mathbb{Z}_+$ be such that $x = \mathrm{gcf}(y, w)$. Thus by (ii) and Lemma 7.2

$$\begin{aligned}
f(x, \mathrm{lcm}(y, w)) &= \prod_{p\in\mathbb{P}} f(x, xp^{\mathrm{lcm}(y,w)(p)-x(p)}) \\
&= \prod_{p\in\mathbb{P}} f(x, xp^{y(p)-x(p)}) f(x, xp^{w(p)-x(p)}) \\
&= \Big[\prod_{p\in\mathbb{P}} f(x, xp^{y(p)-x(p)})\Big]\Big[\prod_{p\in\mathbb{P}} f(x, xp^{w(p)-x(p)})\Big] \\
&= f(x, y) f(x, w).
\end{aligned}$$

Thus f is semifactorable. $\square$

The following theorem presents a characterization of a semifactorable function, which, on the necessity part, depends heavily on the characterization presented above in Theorem 7.5.

Theorem 7.6. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is semifactorable if and only if*

(i) $\forall\, x \in \mathbb{Z}_+ : f(x, x) = 1,$

(ii) $\forall\, x, y, w \in \mathbb{Z}_+ : \Big[x \trianglelefteq y \text{ and } x \trianglelefteq w\Big]$
$\qquad \Rightarrow f(x, \mathrm{gcf}(y, w)) f(x, \mathrm{lcm}(y, w)) = f(x, y) f(x, w).$

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. Let us first assume that f is semifactorable. (i) By the semifactorability of f

$$\forall\, x \in \mathbb{Z}_+ : f(x, x) = 1.$$

(ii) Let $x, y, w \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$ and $x \trianglelefteq w$. Let us note that

$$\begin{aligned}
\forall\, p \in \mathbb{P} :& \Big[\min\{y(p), w(p)\} = y(p) \text{ and } \max\{y(p), w(p)\} = w(p)\Big] \\
\text{or }& \Big[\min\{y(p), w(p)\} = w(p) \text{ and } \max\{y(p), w(p)\} = y(p)\Big],
\end{aligned}$$

where $y = \prod_{p\in\mathbb{P}} p^{y(p)}$ and $w = \prod_{p\in\mathbb{P}} p^{w(p)}$ Let us specifically note that if $\min\{y(p), w(p)\} = y(p)$ and $\max\{y(p), w(p)\} = w(p)$, then

$$\begin{aligned}
&f(x, xp^{\min\{y(p),w(p)\}-x(p)}) f(x, xp^{\max\{y(p),w(p)\}-x(p)}) \\
&\qquad = f(x, xp^{y(p)-x(p)}) f(x, xp^{w(p)-x(p)}),
\end{aligned}$$

where $x = \prod_{p\in\mathbb{P}} p^{x(p)}$. Correspondingly, if $\min\{y(p), w(p)\} = w(p)$ and $\max\{y(p), w(p)\} = y(p)$, then

$$\begin{aligned}
&f(x, xp^{\min\{y(p),w(p)\}-x(p)}) f(x, xp^{\max\{y(p),w(p)\}-x(p)}) \\
&\qquad = f(x, xp^{w(p)-x(p)}) f(x, xp^{y(p)-x(p)}).
\end{aligned}$$

Thus

$$\begin{aligned}\forall\, p \in \mathbb{P} : f(x, xp^{\min\{y(p),w(p)\}-x(p)})&f(x, xp^{\max\{y(p),w(p)\}-x(p)})\\ &= f(x, xp^{y(p)-x(p)})f(x, xp^{w(p)-x(p)}),\end{aligned}$$

and therefore

$$\begin{aligned}\forall\, p \in \mathbb{P} : f(x, xp^{\operatorname{gcf}(y,w)(p)-x(p)})&f(x, xp^{\operatorname{lcm}(y,w)(p)-x(p)})\\ &= f(x, xp^{y(p)-x(p)})f(x, xp^{w(p)-x(p)}).\end{aligned}$$

Thus by Theorem 7.5

$$\begin{aligned}
f(x, \operatorname{gcf}(y, w))&f(x, \operatorname{lcm}(y, w))\\
&= \Big[\prod_{p\in\mathbb{P}} f(x, xp^{\operatorname{gcf}(y,w)(p)-x(p)})\Big]\Big[\prod_{p\in\mathbb{P}} f(x, xp^{\operatorname{lcm}(y,w)(p)-x(p)})\Big]\\
&= \prod_{p\in\mathbb{P}} f(x, xp^{\operatorname{gcf}(y,w)(p)-x(p)})f(x, xp^{\operatorname{lcm}(y,w)(p)-x(p)})\\
&= \prod_{p\in\mathbb{P}} f(x, xp^{y(p)-x(p)})f(x, xp^{w(p)-x(p)})\\
&= \Big[\prod_{p\in\mathbb{P}} f(x, xp^{y(p)-x(p)})\Big]\Big[\prod_{p\in\mathbb{P}} f(x, xp^{w(p)-x(p)})\Big]\\
&= f(x, y)f(x, w).
\end{aligned}$$

Thus (i) and (ii) hold.

Let us next assume that (i) and (ii) hold. Let $x, y, w \in \mathbb{Z}_+$ be such that $x = \operatorname{gcf}(y, w)$. Then $x \trianglelefteq y$ and $x \trianglelefteq w$, and therefore by (i) and (ii)

$$f(x, \operatorname{lcm}(y, w)) = f(x, \operatorname{gcf}(y, w))f(x, \operatorname{lcm}(y, w)) = f(x, y)f(x, w).$$

Thus f is semifactorable. □

Remark. The condition (ii) of Theorem 7.6 is a generalization of the notion of semimultiplicativity of an arithmetic function (see e.g. [19], [31], [32], [46]).

Theorem 7.7. *If $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ are semifactorable, then $f * g$ is semifactorable.*

Proof. Let $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be semifactorable. (i) Let $x \in \mathbb{Z}_+$. Then by the semifactorability of f and g

$$(f * g)(x, x) = \sum_{\substack{x\trianglelefteq z\trianglelefteq x\\ zw=xx}} f(x, z)g(x, w) = f(x, x)g(x, x) = 1 \cdot 1 = 1.$$

(ii) Let $x, y, w \in \mathbb{Z}_+$ be such that $x = \operatorname{gcf}(y, w)$. Let us note that by Lemma 5.9 the function

$$\beta_1 : [\,x, y\,] \times [\,x, w\,] \to [\,x, \operatorname{lcm}(y, w)\,] : \beta_1(u, v) = \operatorname{lcm}(u, v)$$

is one-to-one and onto. Thus by Lemmas 5.9, 5.10, and 5.8 and the semifactorability of f and g

$$\begin{aligned}
(f*g)(x,\operatorname{lcm}(y,w)) &= \sum_{\substack{x\trianglelefteq q\trianglelefteq \operatorname{lcm}(y,w)\\ qr=x\operatorname{lcm}(y,w)}} f(x,q)g(x,r)\\
&\overset{(1)}{=} \sum_{\substack{x\trianglelefteq u\trianglelefteq y,\, x\trianglelefteq s\trianglelefteq w\\ x\trianglelefteq v\trianglelefteq y,\, x\trianglelefteq t\trianglelefteq w\\ \operatorname{lcm}(u,s)\operatorname{lcm}(v,t)=x\operatorname{lcm}(y,w)}} f(x,\operatorname{lcm}(u,s))g(x,\operatorname{lcm}(v,t))\\
&\overset{(2)}{=} \sum_{\substack{x\trianglelefteq u\trianglelefteq y\\ uv=xy\\ x\trianglelefteq s\trianglelefteq w\\ st=xw}} f(x,\operatorname{lcm}(u,s))g(x,\operatorname{lcm}(v,t))\\
&\overset{(3)}{=} \sum_{\substack{x\trianglelefteq u\trianglelefteq y\\ uv=xy\\ x\trianglelefteq s\trianglelefteq w\\ st=xw}} f(x,u)f(x,s)g(x,v)g(x,t)\\
&= \Big[\sum_{\substack{x\trianglelefteq u\trianglelefteq y\\ uv=xy}} f(x,u)g(x,v)\Big]\Big[\sum_{\substack{x\trianglelefteq s\trianglelefteq w\\ st=xw}} f(x,s)g(x,t)\Big]\\
&= (f*g)(x,y)(f*g)(x,w),
\end{aligned}$$

where (1) means "by Lemma 5.9",
(2) means "by Lemma 5.10",
(3) means "by Lemma 5.8 and the semifactorability of f and g".

By (i) and (ii) $f*g$ is semifactorable. $\square$

Theorem 7.8. *If $f\in\mathbb{I}[\mathbb{Z}_+,\trianglelefteq]$ is semifactorable, then f^{*-1} is semifactorable.*

Proof. Let $f\in\mathbb{I}[\mathbb{Z}_+,\trianglelefteq]$ be semifactorable. (i) By the semifactorability of f and Theorem 5.21

$$\forall\, x\in\mathbb{Z}_+ : f^{*-1}(x,x)=1.$$

(ii) Let $x,y,w\in\mathbb{Z}_+$ be such that $x=\operatorname{gcf}(y,w)$. Let us note that by Lemma 5.9 the function

$$\beta_1 : [\,x,y\,]\times[\,x,w\,]\to[\,x,\operatorname{lcm}(y,w)\,] : \beta_1(u,v)=\operatorname{lcm}(u,v)$$

is one-to-one and onto and use induction on the interval $[\,x,\operatorname{lcm}(y,w)\,]$. If $x=\operatorname{lcm}(y,w)$, then $x=y$ and $x=w$, since $\operatorname{lcm}(x,x)=x$ and the function β_1 is one-to-one, and therefore by (i)

$$f^{*-1}(x,\operatorname{lcm}(y,w))=1=f^{*-1}(x,y)f^{*-1}(x,w).$$

Let $x\vartriangleleft\operatorname{lcm}(y,w)$. Let us assume that if $u,s\in\mathbb{Z}_+$ are such that $x\trianglelefteq\operatorname{lcm}(u,s)\vartriangleleft\operatorname{lcm}(y,w)$, then

$$f^{*-1}(x,\operatorname{lcm}(u,s))=f^{*-1}(x,u)f^{*-1}(x,s).$$

Let us note that since the function β_1 is one-to-one, it follows that $\mathrm{lcm}(u,s) = \mathrm{lcm}(y,w)$ if and only if $u = y$ and $s = w$, and correspondingly $\mathrm{lcm}(v,t) = x$ if and only if $v = x$ and $t = x$. Let us also note that since $x \lhd \mathrm{lcm}(y,w)$, it follows that $x \neq y$ or $x \neq w$. Thus by the semifactorability of f, Theorem 5.21, Lemmas 5.3, 5.9, 5.10, and 5.8, and the induction hypothesis

$$\begin{aligned}
f^{*-1}(x,\mathrm{lcm}(y,w)) &\overset{(1)}{=} -\sum_{\substack{x\unlhd q\lhd \mathrm{lcm}(y,w)\\ qr=x\,\mathrm{lcm}(y,w)}} f^{*-1}(x,q)f(x,r)\\
&\overset{(2)}{=} -\sum_{\substack{x\unlhd u\unlhd y,\, x\unlhd s\unlhd w\\ x\unlhd v\unlhd y,\, x\unlhd t\unlhd w\\ \mathrm{lcm}(u,s)\,\mathrm{lcm}(v,t)=x\,\mathrm{lcm}(y,w)\\ \langle u,s\rangle\neq\langle y,w\rangle,\, \langle v,t\rangle\neq\langle x,x\rangle}} f^{*-1}(x,\mathrm{lcm}(u,s))f(x,\mathrm{lcm}(v,t))\\
&\overset{(3)}{=} -\sum_{\substack{x\unlhd u\unlhd y\\ uv=xy\\ x\unlhd s\unlhd w\\ st=xw\\ \langle u,s\rangle\neq\langle y,w\rangle\\ \langle v,t\rangle\neq\langle x,x\rangle}} f^{*-1}(x,\mathrm{lcm}(u,s))f(x,\mathrm{lcm}(v,t))\\
&\overset{(4)}{=} -\sum_{\substack{x\unlhd u\unlhd y\\ uv=xy\\ x\unlhd s\unlhd w\\ st=xw\\ \langle u,s\rangle\neq\langle y,w\rangle\\ \langle v,t\rangle\neq\langle x,x\rangle}} f^{*-1}(x,u)f^{*-1}(x,s)f(x,v)f(x,t)\\
&= -\sum_{\substack{x\unlhd u\unlhd y\\ uv=xy\\ x\unlhd s\unlhd w\\ st=xw}} f^{*-1}(x,u)f^{*-1}(x,s)f(x,v)f(x,t)\\
&\quad+\sum_{\substack{x\unlhd u\unlhd y\\ uv=xy\\ x\unlhd s\unlhd w\\ st=xw\\ \langle u,s\rangle=\langle y,w\rangle\\ \langle v,t\rangle=\langle x,x\rangle}} f^{*-1}(x,u)f^{*-1}(x,s)f(x,v)f(x,t)\\
&= -\Big[\sum_{\substack{x\unlhd u\unlhd y\\ uv=xy}} f^{*-1}(x,u)f(x,v)\Big]\Big[\sum_{\substack{x\unlhd s\unlhd w\\ st=xw}} f^{*-1}(x,s)f(x,t)\Big]\\
&\quad+ f^{*-1}(x,y)f^{*-1}(x,w)f(x,x)f(x,x)\\
&= -\Big[(f^{*-1}*f)(x,y)(f^{*-1}*f)(x,w)\Big]\\
&\quad+ f^{*-1}(x,y)f^{*-1}(x,w)\\
&= -\Big[\delta(x,y)\delta(x,w)\Big] + f^{*-1}(x,y)f^{*-1}(x,w)\\
&= f^{*-1}(x,y)f^{*-1}(x,w),
\end{aligned}$$

where (1) means "by the semifactorability of f and Theorem 5.21",
(2) means "by Lemmas 5.3 and 5.9",
(3) means "by Lemma 5.10",
(4) means "by the semifactorability of f, Lemma 5.8, and the induction hypothesis".

By (i) and (ii) f^{*-1} is semifactorable. □

Theorem 7.9. *The Möbius function $\mu \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is semifactorable.*

Proof. Follows by Theorems 7.1 and 7.8. □

Theorem 7.10. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is semifactorable if and only if its divisibility order summatory function is semifactorable.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, and let $F \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be the divisibility order summatory function of the function f. Let us first assume that f is semifactorable. Then by Theorems 7.1 and 7.7 $f * \zeta$ is semifactorable, and therefore by Theorem 5.15 F is semifactorable.

Let us next assume that F is semifactorable. Then by Theorems 7.9 and 7.7 $F * \mu$ is semifactorable, and therefore by Theorem 5.24 f is semifactorable. □

Theorem 7.11. *If $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ are semifactorable, then $f *_C g$ is semifactorable.*

Proof. Let $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be semifactorable. (i) Let $x \in \mathbb{Z}_+$. Then by the semifactorability of f and g

$$(f *_C g)(x,x) = \sum_{\substack{\mathrm{gcf}(z,w)=x \\ \mathrm{lcm}(z,w)=x}} f(x,z)g(x,w) = f(x,x)g(x,x) = 1 \cdot 1 = 1.$$

(ii) Let $x, y, w \in \mathbb{Z}_+$ be such that $x = \mathrm{gcf}(y,w)$. Let us note that by Lemma 5.9 the function

$$\beta_1 : [\,x, y\,] \times [\,x, w\,] \to [\,x, \mathrm{lcm}(y,w)\,] : \beta_1(u,v) = \mathrm{lcm}(u,v)$$

is one-to-one and onto. Thus by Lemmas 5.15, 5.9, 5.10, 5.19, and 5.8 and the semifactorability of f and g

$$\begin{aligned}
(f *_C g)(x, \operatorname{lcm}(y,w)) &= \sum_{\substack{\operatorname{gcf}(q,r)=x \\ \operatorname{lcm}(q,r)=\operatorname{lcm}(y,w)}} f(x,q)g(x,r) \\
&\overset{(1)}{=} \sum_{\substack{x \trianglelefteq q \trianglelefteq \operatorname{lcm}(y,w) \\ qr = x\operatorname{lcm}(y,w) \\ \operatorname{gcf}(q,r)=x \\ \operatorname{lcm}(q,r)=\operatorname{lcm}(y,w)}} f(x,q)g(x,r) \\
&\overset{(2)}{=} \sum_{\substack{x \trianglelefteq u \trianglelefteq y,\, x \trianglelefteq s \trianglelefteq w \\ x \trianglelefteq v \trianglelefteq y,\, x \trianglelefteq t \trianglelefteq w \\ \operatorname{lcm}(u,s)\operatorname{lcm}(v,t) = x\operatorname{lcm}(y,w) \\ \operatorname{gcf}(\operatorname{lcm}(u,s),\operatorname{lcm}(v,t))=\operatorname{lcm}(x,x) \\ \operatorname{lcm}(\operatorname{lcm}(u,s),\operatorname{lcm}(v,t))=\operatorname{lcm}(y,w)}} f(x,\operatorname{lcm}(u,s))g(x,\operatorname{lcm}(v,t)) \\
&\overset{(3)}{=} \sum_{\substack{x \trianglelefteq u \trianglelefteq y \\ uv=xy \\ \operatorname{gcf}(u,v)=x \\ \operatorname{lcm}(u,v)=y \\ x \trianglelefteq s \trianglelefteq w \\ st=xw \\ \operatorname{gcf}(s,t)=x \\ \operatorname{lcm}(s,t)=w}} f(x,\operatorname{lcm}(u,s))g(x,\operatorname{lcm}(v,t)) \\
&\overset{(4)}{=} \sum_{\substack{x \trianglelefteq u \trianglelefteq y \\ uv=xy \\ \operatorname{gcf}(u,v)=x \\ \operatorname{lcm}(u,v)=y \\ x \trianglelefteq s \trianglelefteq w \\ st=xw \\ \operatorname{gcf}(s,t)=x \\ \operatorname{lcm}(s,t)=w}} f(x,u)f(x,s)g(x,v)g(x,t) \\
&= \Big[\sum_{\substack{x \trianglelefteq u \trianglelefteq y \\ uv=xy \\ \operatorname{gcf}(u,v)=x \\ \operatorname{lcm}(u,v)=y}} f(x,u)g(x,v)\Big]\Big[\sum_{\substack{x \trianglelefteq s \trianglelefteq w \\ st=xw \\ \operatorname{gcf}(s,t)=x \\ \operatorname{lcm}(s,t)=w}} f(x,s)g(x,t)\Big] \\
&\overset{(1)}{=} \Big[\sum_{\substack{\operatorname{gcf}(u,v)=x \\ \operatorname{lcm}(u,v)=y}} f(x,u)g(x,v)\Big]\Big[\sum_{\substack{\operatorname{gcf}(s,t)=x \\ \operatorname{lcm}(s,t)=w}} f(x,s)g(x,t)\Big] \\
&= (f *_C g)(x,y)(f *_C g)(x,w),
\end{aligned}$$

where (1) means “by Lemma 5.15”,
(2) means “by Lemma 5.9”,
(3) means “by Lemmas 5.10 and 5.19”,
(4) means “by Lemma 5.8 and the semifactorability of f and g”.

By (i) and (ii) $f *_C g$ is semifactorable. $\square$

Theorem 7.12. *If $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is semifactorable, then $f^{*_C -1}$ is semifactorable.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be semifactorable. (i) By the semifactorability of f and Theorem 5.39

$$\forall\, x \in \mathbb{Z}_+ : f^{*_C -1}(x,x) = 1.$$

(ii) Let $x, y, w \in \mathbb{Z}_+$ be such that $x = \mathrm{gcf}(y,w)$. Let us note that by Lemma 5.9 the function

$$\beta_1 : [\,x, y\,] \times [\,x, w\,] \to [\,x, \mathrm{lcm}(y,w)\,] : \beta_1(u,v) = \mathrm{lcm}(u,v)$$

is one-to-one and onto and use induction on the interval $[\,x, \mathrm{lcm}(y,w)\,]$. If $x = \mathrm{lcm}(y,w)$, then $x = y$ and $x = w$, since $\mathrm{lcm}(x,x) = x$ and the function β_1 is one-to-one, and therefore by (i)

$$f^{*_C -1}(x, \mathrm{lcm}(y,w)) = 1 = f^{*_C -1}(x,y) f^{*_C -1}(x,w).$$

Let $x \vartriangleleft \mathrm{lcm}(y,w)$. Let us assume that if $u, s \in \mathbb{Z}_+$ are such that $x \trianglelefteq \mathrm{lcm}(u,s) \vartriangleleft \mathrm{lcm}(y,w)$, then

$$f^{*_C -1}(x, \mathrm{lcm}(u,s)) = f^{*_C -1}(x,u) f^{*_C -1}(x,s).$$

Let us note that since the function β_1 is one-to-one, it follows that $\mathrm{lcm}(u,s) = \mathrm{lcm}(y,w)$ if and only if $u = y$ and $s = w$, and correspondingly $\mathrm{lcm}(v,t) = x$ if and only if $v = x$ and $t = x$. Let us also note that since $x \vartriangleleft \mathrm{lcm}(y,w)$, it follows that $x \neq y$ or $x \neq w$. Thus by the semifactorability of f, Theorem 5.39, Lemmas 5.15, 5.3, 5.9, 5.10, 5.19, and 5.8, and the induction hypothesis

$$\begin{aligned}
&f^{*_C -1}(x, \mathrm{lcm}(y,w)) \\
&\quad\overset{(1)}{=} -\sum_{\substack{\mathrm{gcf}(q,r)=x \\ \mathrm{lcm}(q,r)=\mathrm{lcm}(y,w) \\ q \neq \mathrm{lcm}(y,w)}} f^{*_C -1}(x,q) f(x,r) \\
&\quad\overset{(2)}{=} -\sum_{\substack{x \trianglelefteq q \vartriangleleft \mathrm{lcm}(y,w) \\ qr = x\,\mathrm{lcm}(y,w) \\ \mathrm{gcf}(q,r)=x \\ \mathrm{lcm}(q,r)=\mathrm{lcm}(y,w)}} f^{*_C -1}(x,q) f(x,r) \\
&\quad\overset{(3)}{=} -\sum_{\substack{x \trianglelefteq u \trianglelefteq y,\, x \trianglelefteq s \trianglelefteq w \\ x \trianglelefteq v \trianglelefteq y,\, x \trianglelefteq t \trianglelefteq w \\ \mathrm{lcm}(u,s)\,\mathrm{lcm}(v,t) = x\,\mathrm{lcm}(y,w) \\ \mathrm{gcf}(\mathrm{lcm}(u,s),\mathrm{lcm}(v,t))=\mathrm{lcm}(x,x) \\ \mathrm{lcm}(\mathrm{lcm}(u,s),\mathrm{lcm}(v,t))=\mathrm{lcm}(y,w) \\ \langle u,s\rangle \neq \langle y,w\rangle,\, \langle v,t\rangle \neq \langle x,x\rangle}} f^{*_C -1}(x, \mathrm{lcm}(u,s)) f(x, \mathrm{lcm}(v,t))
\end{aligned}$$

$$
\begin{aligned}
&\overset{(4)}{=} -\sum_{\substack{x\trianglelefteq u\trianglelefteq y\\ uv=xy\\ \mathrm{gcf}(u,v)=x\\ \mathrm{lcm}(u,v)=y\\ x\trianglelefteq s\trianglelefteq w\\ st=xw\\ \mathrm{gcf}(s,t)=x\\ \mathrm{lcm}(s,t)=w\\ \langle u,s\rangle\neq\langle y,w\rangle\\ \langle v,t\rangle\neq\langle x,x\rangle}} f^{*_C-1}(x,\mathrm{lcm}(u,s))f(x,\mathrm{lcm}(v,t))\\
&\overset{(5)}{=} -\sum_{\substack{x\trianglelefteq u\trianglelefteq y\\ uv=xy\\ \mathrm{gcf}(u,v)=x\\ \mathrm{lcm}(u,v)=y\\ x\trianglelefteq s\trianglelefteq w\\ st=xw\\ \mathrm{gcf}(s,t)=x\\ \mathrm{lcm}(s,t)=w\\ \langle u,s\rangle\neq\langle y,w\rangle\\ \langle v,t\rangle\neq\langle x,x\rangle}} f^{*_C-1}(x,u)f^{*_C-1}(x,s)f(x,v)f(x,t)\\
&= -\sum_{\substack{x\trianglelefteq u\trianglelefteq y\\ uv=xy\\ \mathrm{gcf}(u,v)=x\\ \mathrm{lcm}(u,v)=y\\ x\trianglelefteq s\trianglelefteq w\\ st=xw\\ \mathrm{gcf}(s,t)=x\\ \mathrm{lcm}(s,t)=w}} f^{*_C-1}(x,u)f^{*_C-1}(x,s)f(x,v)f(x,t)\\
&\quad + \sum_{\substack{x\trianglelefteq u\trianglelefteq y\\ uv=xy\\ \mathrm{gcf}(u,v)=x\\ \mathrm{lcm}(u,v)=y\\ x\trianglelefteq s\trianglelefteq w\\ st=xw\\ \mathrm{gcf}(s,t)=x\\ \mathrm{lcm}(s,t)=w\\ \langle u,s\rangle=\langle y,w\rangle\\ \langle v,t\rangle=\langle x,x\rangle}} f^{*_C-1}(x,u)f^{*_C-1}(x,s)f(x,v)f(x,t)\\
&= -\Big[\sum_{\substack{x\trianglelefteq u\trianglelefteq y\\ uv=xy\\ \mathrm{gcf}(u,v)=x\\ \mathrm{lcm}(u,v)=y}} f^{*_C-1}(x,u)f(x,v)\Big]\Big[\sum_{\substack{x\trianglelefteq s\trianglelefteq w\\ st=xw\\ \mathrm{gcf}(s,t)=x\\ \mathrm{lcm}(s,t)=w}} f^{*_C-1}(x,s)f(x,t)\Big]\\
&\quad + f^{*_C-1}(x,y)f^{*_C-1}(x,w)f(x,x)f(x,x)\\
&\overset{(2)}{=} -\Big[\sum_{\substack{\mathrm{gcf}(u,v)=x\\ \mathrm{lcm}(u,v)=y}} f^{*_C-1}(x,u)f(x,v)\Big]\Big[\sum_{\substack{\mathrm{gcf}(s,t)=x\\ \mathrm{lcm}(s,t)=w}} f^{*_C-1}(x,s)f(x,t)\Big]\\
&\quad + f^{*_C-1}(x,y)f^{*_C-1}(x,w)\\
&= -\Big[(f^{*_C-1} *_C f)(x,y)(f^{*_C-1} *_C f)(x,w)\Big] + f^{*_C-1}(x,y)f^{*_C-1}(x,w)\\
&= -\Big[\delta(x,y)\delta(x,w)\Big] + f^{*_C-1}(x,y)f^{*_C-1}(x,w)\\
&= f^{*_C-1}(x,y)f^{*_C-1}(x,w),
\end{aligned}
$$

where (1) means "by the semifactorability of f and Theorem 5.39",
(2) means "by Lemma 5.15",
(3) means "by Lemmas 5.3 and 5.9",
(4) means "by Lemmas 5.10 and 5.19",
(5) means "by the semifactorability of f, Lemma 5.8, and the induction hypothesis".

By (i) and (ii) $f^{*_C -1}$ is semifactorable. $\square$

Theorem 7.13. *The complementary Möbius function $\mu_c \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is semifactorable.*

Proof. Follows by Theorems 7.1 and 7.12. $\square$

Theorem 7.14. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is semifactorable if and only if its complementary summatory function is semifactorable.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, and let $F \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be the complementary summatory function of the function f. Let us first assume that f is semifactorable. Then by Theorems 7.1 and 7.11 $f *_C \zeta$ is semifactorable, and therefore by Theorem 5.33 F is semifactorable.

Let us next assume that F is semifactorable. Then by Theorems 7.13 and 7.11 $F *_C \mu_c$ is semifactorable, and therefore by Theorem 5.42 f is semifactorable. $\square$

The following theorem presents a characterization of a semifactorable function that is closely related to Theorem 7.2.

Theorem 7.15. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is semifactorable if and only if*

(i) $\forall\, x \in \mathbb{Z}_+ : f(x, x) = 1,$

(ii) $\forall\, g, h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq] : f(g *_C h) = (fg) *_C (fh).$

Proof. Follows by Theorems 7.2 and 5.45. $\square$

The close resemblance between Theorems 3.36 and 7.15 brings up the question whether there exist specific C-convolutions that possess, in a more or less comparable fashion as in the case of completely multiplicative arithmetic functions, a property to discriminate the semifactorable functions from the 'ordinary' arithmetic incidence functions. Let us proceed with the study into this direction and introduce next the notion of a C-discriminative C-convolution.

Definition 7.2. Let $g, h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. The C-convolution of g and h, that is $g *_C h$, is *C-discriminative* if

$$\begin{aligned}\forall\, x, y \in \mathbb{Z}_+ : (g *_C h)(x, xy) &= g(x, x)h(x, xy) + g(x, xy)h(x, x)\\ &\Rightarrow \omega(x, xy) \leq 1.\end{aligned}$$

Remark. The function $\omega \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is defined in Definition 5.6.

Remark. If $g *_C h$ is C-discriminative, then by Theorem 5.35 also $h *_C g$ is C-discriminative.

Theorem 7.16. *The C-convolution $\zeta *_C \zeta$, where $\zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, is C-discriminative.*

Proof. Let $x, y \in \mathbb{Z}_+$ be such that $\omega(x, xy) = n > 1$. Let $u, v \in \mathbb{Z}_+$ be such that $y = uv$, where

$$y = \prod_{p \in \mathbb{P}} p^{y(p)} = \prod_{i=1}^{n} p_i^{y(p_i)}, \quad u = \prod_{i=1}^{n-1} p_i^{y(p_i)}, \quad v = p_n^{y(p_n)}.$$

Then $\operatorname{gcf}(xu, xv) = x$, $\operatorname{lcm}(xu, xv) = xy$, $xu \neq x$, $xv \neq xy$, $xu \neq xy$, and $xv \neq x$. Thus by Lemmas 5.15 and 5.3

$$\begin{aligned}(\zeta *_C \zeta)(x, xy) &= \sum_{\substack{\operatorname{gcf}(z,w)=x \\ \operatorname{lcm}(z,w)=xy}} \zeta(x, z)\zeta(x, w)\\ &= \zeta(x, x)\zeta(x, xy) + \zeta(x, xy)\zeta(x, x) + \sum_{\substack{\operatorname{gcf}(z,w)=x \\ \operatorname{lcm}(z,w)=xy \\ z \neq x,\, w \neq xy \\ z \neq xy,\, w \neq x}} \zeta(x, z)\zeta(x, w),\end{aligned}$$

where

$$\sum_{\substack{\operatorname{gcf}(z,w)=x \\ \operatorname{lcm}(z,w)=xy \\ z \neq x,\, w \neq xy \\ z \neq xy,\, w \neq x}} \zeta(x, z)\zeta(x, w) > 0,$$

and therefore $(\zeta *_C \zeta)(x, xy) \neq \zeta(x, x)\zeta(x, xy) + \zeta(x, xy)\zeta(x, x)$. Thus by the contraposition principle

$$\begin{aligned}\forall\, x, y \in \mathbb{Z}_+ : (\zeta *_C \zeta)(x, xy) &= \zeta(x, x)\zeta(x, xy) + \zeta(x, xy)\zeta(x, x)\\ &\Rightarrow \omega(x, xy) \leq 1.\end{aligned}$$

and therefore by Definition 7.2 $\zeta *_C \zeta$ is C-discriminative. □

Remark. The C-convolution $\zeta *_C \zeta$, where $\zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, is a generalization of the function $\theta \in \mathbb{A}$ (see Definition 3.6).

Lemma 7.3. *Let* $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. *If* $x, y, z, w \in \mathbb{Z}_+$ *are such that* $\mathrm{gcf}(z, w) = x$, $\mathrm{lcm}(z, w) = y$, $z \neq x$, $w \neq y$, $z \neq y$, *and* $w \neq x$, *then* $\omega(x, z) < \omega(x, y)$ *and* $\omega(x, w) < \omega(x, y)$.

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, and let $x, y, z, w \in \mathbb{Z}_+$, where

$$x = \prod_{p \in \mathbb{P}} p^{x(p)}, \quad z = \prod_{p \in \mathbb{P}} p^{z(p)}, \quad w = \prod_{p \in \mathbb{P}} p^{w(p)}, \quad y = \prod_{p \in \mathbb{P}} p^{y(p)},$$

be such that $\mathrm{gcf}(z, w) = x$, $\mathrm{lcm}(z, w) = y$, $z \neq x$, $w \neq y$, $z \neq y$, and $w \neq x$. Then by Lemmas 5.15 and 5.3 $x \lhd z \lhd y$ and $x \lhd w \lhd y$, and therefore by Definition 5.6 $\omega(x, z) \leq \omega(x, y)$ and $\omega(x, w) \leq \omega(x, y)$. Let us note that since $x \trianglelefteq z$, $x \trianglelefteq w$, and $\mathrm{gcf}(x, x) = \mathrm{gcf}(z, w)$, it follows by Lemma 6.3 that

$$\forall\, p \in \mathbb{P} : x(p) = w(p) \text{ or } x(p) = z(p).$$

Let us assume that $\omega(x, z) = \omega(x, y)$. Then by Definition 5.6 it follows that

$$\forall\, p \in \mathbb{P} : x(p) \neq y(p) \Rightarrow x(p) \neq z(p).$$

Since $x \trianglelefteq w \trianglelefteq y$, it follows that

$$\forall\, p \in \mathbb{P} : x(p) = y(p) \Rightarrow x(p) = w(p).$$

Thus

$$\forall\, p \in \mathbb{P} : x(p) = w(p),$$

and therefore $w = x$. This contradicts the fact that $w \neq x$, and therefore $\omega(x, z) \neq \omega(x, y)$. Thus $\omega(x, z) < \omega(x, y)$. Correspondingly $\omega(x, w) \neq \omega(x, y)$, and therefore $\omega(x, w) < \omega(x, y)$. □

The following theorems present characterizations of a semifactorable function.

Theorem 7.17. *A function* $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ *is semifactorable if and only if*

(i) $\forall\, x \in \mathbb{Z}_+ : f(x, x) = 1$,

(ii) $\exists\, g, h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq] : f(g *_C h) = (fg) *_C (fh)$, *where* $g *_C h$ *is* C*-discriminative.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. Let us first assume that f is semifactorable. (i) By the semifactorability of f

$$\forall\, x \in \mathbb{Z}_+ : f(x, x) = 1.$$

(ii) By Theorem 7.15 $f(\zeta *_C \zeta) = (f\zeta) *_C (f\zeta)$, where (by Theorem 7.16) $\zeta *_C \zeta$ is C-discriminative. Thus (i) and (ii) hold.

Let us next assume that (i) and (ii) hold. Let $x, y \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$, where $x = \prod_{p\in\mathbb{P}} p^{x(p)}$ and $y = \prod_{p\in\mathbb{P}} p^{y(p)}$. Let us use induction on $\omega(x,y) \in \mathbb{N}$, where $\omega(x,y)$ stands for the number of distinct prime factors of y/x (see Definition 5.6), to show that

$$f(x,y) = \prod_{p\in\mathbb{P}} f(x, xp^{y(p)-x(p)}).$$

If $\omega(x,y) = 0$, then $x(p) = y(p)$ for all $p \in \mathbb{P}$, and therefore by (i)

$$f(x,y) = f(x,x) = 1 = \prod_{p\in\mathbb{P}} f(x, xp^{y(p)-x(p)}).$$

If $\omega(x,y) = 1$, then $y = xp^n$, where $p \in \mathbb{P}$, $n \in \mathbb{Z}_+$, and $y(p) = x(p) + n$, and therefore by (i)

$$f(x,y) = f(x, xp^n) = \prod_{p\in\mathbb{P}} f(x, xp^{y(p)-x(p)}).$$

Let $\omega(x,y) = n > 1$, and let us assume that

$$\forall\, u \in \mathbb{Z}_+ : (x \trianglelefteq u \trianglelefteq y \text{ and } \omega(x,u) < n) \Rightarrow f(x,u) = \prod_{p\in\mathbb{P}} f(x, xp^{u(p)-x(p)}),$$

where $u = \prod_{p\in\mathbb{P}} p^{u(p)}$. Let us assume that $g, h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ satisfy the condition (ii), i.e. $g *_C h$ is C-discriminative and $f(g *_C h) = (fg) *_C (fh)$. Then by Lemmas 5.15 and 5.3, (i), Lemma 7.3, the induction hypothesis, and Lemma 7.2

$$\begin{aligned}
[(fg) *_C (fh)](x,y) &= \sum_{\substack{\mathrm{gcf}(z,w)=x\\ \mathrm{lcm}(z,w)=y}} (fg)(x,z)(fh)(x,w)\\
&= \sum_{\substack{\mathrm{gcf}(z,w)=x\\ \mathrm{lcm}(z,w)=y}} f(x,z)g(x,z)f(x,w)h(x,w)\\
&\overset{(1)}{=} f(x,x)g(x,x)f(x,y)h(x,y) + f(x,y)g(x,y)f(x,x)h(x,x)\\
&\quad + \sum_{\substack{\mathrm{gcf}(z,w)=x\\ \mathrm{lcm}(z,w)=y\\ z\neq x,\, w\neq y\\ z\neq y,\, w\neq x}} f(x,z)g(x,z)f(x,w)h(x,w)\\
&\overset{(2)}{=} f(x,y)[g(x,x)h(x,y) + g(x,y)h(x,x)]\\
&\quad + \sum_{\substack{\mathrm{gcf}(z,w)=x\\ \mathrm{lcm}(z,w)=y\\ z\neq x,\, w\neq y\\ z\neq y,\, w\neq x}} \Big[\Big[\prod_{p\in\mathbb{P}} f(x, xp^{z(p)-x(p)})\Big] g(x,z)\\
&\qquad\qquad \times \Big[\prod_{p\in\mathbb{P}} f(x, xp^{w(p)-x(p)})\Big] h(x,w)\Big]
\end{aligned}$$

$$
\begin{aligned}
&= f(x,y)[g(x,x)h(x,y) + g(x,y)h(x,x)] \\
&\quad + \sum_{\substack{\mathrm{gcf}(z,w)=x \\ \mathrm{lcm}(z,w)=y \\ z\neq x,\, w\neq y \\ z\neq y,\, w\neq x}} \Big[\Big[\prod_{p\in\mathbb{P}} f(x, xp^{z(p)-x(p)}) f(x, xp^{w(p)-x(p)})\Big] \\
&\qquad\qquad\qquad \times g(x,z)h(x,w)\Big] \\
&\overset{(3)}{=} f(x,y)[g(x,x)h(x,y) + g(x,y)h(x,x)] \\
&\quad + \sum_{\substack{\mathrm{gcf}(z,w)=x \\ \mathrm{lcm}(z,w)=y \\ z\neq x,\, w\neq y \\ z\neq y,\, w\neq x}} \Big[\prod_{p\in\mathbb{P}} f(x, xp^{\mathrm{lcm}(z,w)(p)-x(p)})\Big] g(x,z)h(x,w) \\
&= f(x,y)[g(x,x)h(x,y) + g(x,y)h(x,x)] \\
&\quad + \sum_{\substack{\mathrm{gcf}(z,w)=x \\ \mathrm{lcm}(z,w)=y \\ z\neq x,\, w\neq y \\ z\neq y,\, w\neq x}} \Big[\prod_{p\in\mathbb{P}} f(x, xp^{y(p)-x(p)})\Big] g(x,z)h(x,w) \\
&= f(x,y)[g(x,x)h(x,y) + g(x,y)h(x,x)] \\
&\quad + \prod_{p\in\mathbb{P}} f(x, xp^{y(p)-x(p)}) \sum_{\substack{\mathrm{gcf}(z,w)=x \\ \mathrm{lcm}(z,w)=y \\ z\neq x,\, w\neq y \\ z\neq y,\, w\neq x}} g(x,z)h(x,w),
\end{aligned}
$$

where (1) means "by Lemmas 5.15 and 5.3",
(2) means "by (i), Lemma 7.3, and the induction hypothesis",
(3) means "by Lemma 7.2.

On the other hand, by Lemmas 5.15 and 5.3

$$
\begin{aligned}
[f(g *_C h)](x,y) &= f(x,y)[g(x,x)h(x,y) + g(x,y)h(x,x)] \\
&\quad + f(x,y) \sum_{\substack{\mathrm{gcf}(z,w)=x \\ \mathrm{lcm}(z,w)=y \\ z\neq x,\, w\neq y \\ z\neq y,\, w\neq x}} g(x,z)h(x,w),
\end{aligned}
$$

and therefore, since $f(g *_C h) = (fg) *_C (fh)$, it follows that

$$
f(x,y) \sum_{\substack{\mathrm{gcf}(z,w)=x \\ \mathrm{lcm}(z,w)=y \\ z\neq x,\, w\neq y \\ z\neq y,\, w\neq x}} g(x,z)h(x,w) = \prod_{p\in\mathbb{P}} f(x, xp^{y(p)-x(p)}) \sum_{\substack{\mathrm{gcf}(z,w)=x \\ \mathrm{lcm}(z,w)=y \\ z\neq x,\, w\neq y \\ z\neq y,\, w\neq x}} g(x,z)h(x,w).
$$

Since $\omega(x,y) = n > 1$ and $g * h$ is C-discriminative, it follows that

$$
(g *_C h)(x,y) \neq g(x,x)h(x,y) + g(x,y)h(x,x).
$$

On the other hand (by Lemmas 5.15 and 5.3)

$$(g *_C h)(x, y) = g(x, x)h(x, y) + g(x, y)h(x, x) + \sum_{\substack{\mathrm{gcf}(z,w)=x \\ \mathrm{lcm}(z,w)=y \\ z\neq x,\, w\neq y \\ z\neq y,\, w\neq x}} g(x, z)h(x, w),$$

and therefore

$$\sum_{\substack{\mathrm{gcf}(z,w)=x \\ \mathrm{lcm}(z,w)=y \\ z\neq x,\, w\neq y \\ z\neq y,\, w\neq x}} g(x, z)h(x, w) \neq 0.$$

Thus

$$f(x, y) = \prod_{p\in\mathbb{P}} f(x, xp^{y(p)-x(p)}),$$

and therefore

$$\forall\, x, y \in \mathbb{Z}_+ : x \trianglelefteq y \Rightarrow f(x, y) = \prod_{p\in\mathbb{P}} f(x, xp^{y(p)-x(p)}),$$

where $x = \prod_{p\in\mathbb{P}} p^{x(p)}$ and $y = \prod_{p\in\mathbb{P}} p^{y(p)}$. Thus by Theorem 7.5 f is semifactorable. $\square$

Theorem 7.18. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is semifactorable if and only if*

(i) $\forall\, x \in \mathbb{Z}_+ : f(x, x) = 1$,

(ii) $f(\zeta *_C \zeta) = f *_C f$.

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. Let us first assume that f is semifactorable. Let us note that by Theorem 5.9 (ii) is equivalent to $f(\zeta *_C \zeta) = (f\zeta) *_C (f\zeta)$. Thus by Theorem 7.15 (i) and (ii) hold.

Let us next assume that (i) and (ii) hold. Since by Theorem 7.16 $\zeta *_C \zeta$ is C-discriminative and by Theorem 5.9 $f(\zeta *_C \zeta) = (f\zeta) *_C (f\zeta)$, it follows by Theorem 7.17 that f is semifactorable. $\square$

Theorem 7.19. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is semifactorable if and only if*

(i) $\forall\, x \in \mathbb{Z}_+ : f(x, x) = 1$,

(ii) $\forall\, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq] : f(g *_C g) = (fg) *_C (fg)$.

Proof. Follows by Theorems 7.15 and 7.18. $\square$

Let us next introduce a weaker notion of semifactorability called partial semifactorability which is based on the characterization of a semifactorable arithmetic incidence function presented in Theorem 7.5. The primary reason to introduce the concept of partial semifactorability becomes clear in the following discussion.

Definition 7.3. A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is *partially semifactorable* if

(i) $\forall\, x \in \mathbb{Z}_+ : f(x,x) = 1$,

(ii) $\forall\, x, y \in \mathbb{Z}_+ : \omega(x,y)$ is odd $\Rightarrow f(x,y) = \prod_{p\in\mathbb{P}} f(x, xp^{y(p)-x(p)})$, where $x = \prod_{p\in\mathbb{P}} p^{x(p)}$ and $y = \prod_{p\in\mathbb{P}} p^{y(p)}$.

Remark. If $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ satisfies the condition (i) in Definition 7.3, then

$$\forall\, x, y \in \mathbb{Z}_+ : \omega(x,y) = 1 \Rightarrow f(x,y) = \prod_{p\in\mathbb{P}} f(x, xp^{y(p)-x(p)}).$$

If $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is semifactorable, then by Definition 5.6 and Theorem 7.5 it is also partially semifactorable.

Definition 7.4. The function $\zeta_{\omega_{odd}} \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is defined as follows:

$$\zeta_{\omega_{odd}} : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C} : \zeta_{\omega_{odd}}(x,y) = \begin{cases} 1 & \text{if } x = y, \\ 1 & \text{if } \omega(x,y) \text{ is odd}, \\ 0 & \text{otherwise.} \end{cases}$$

The function $\zeta_{\omega_{odd}} \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is an example of a partially semifactorable function that is not semifactorable.

Definition 7.5. Let $g, h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. The C-convolution of g and h, that is $g *_C h$, is *partially C-discriminative* if

$$\begin{aligned} \forall\, x, y \in \mathbb{Z}_+ : (g *_C h)(x, xy) &= g(x,x)h(x,xy) + g(x,xy)h(x,x) \\ &\Rightarrow \omega(x,xy) = 0 \text{ or } \omega(x,xy) \text{ is odd.} \end{aligned}$$

Remark. $\omega(x,xy) = 0$ if and only if $y = 1$.

Remark. From the Definition 7.2 it follows that if $g *_C h$ is C-discriminative, then it is also partially C-discriminative.

Remark. If $g *_C h$ is partially C-discriminative, then by Theorem 5.35 also $h *_C g$ is partially C-discriminative.

The proof of the following theorem explains the motivation for the use of the 'odd omega' property in Definitions 7.3 and 7.5.

Theorem 7.20. *The C-convolution $\zeta *_C \mu_c$, where $\zeta, \mu_c \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, is not C-discriminative, but it is partially C-discriminative.*

Proof. Let $x, y \in \mathbb{Z}_+$, where $x = \prod_{p\in\mathbb{P}} p^{x(p)}$ and $y = \prod_{p\in\mathbb{P}} p^{y(p)}$. Then by Theorems 7.13, 7.5, and 5.48

$$\begin{aligned} \zeta(x,x)\mu_c(x,xy) + \zeta(x,xy)\mu_c(x,x) &= \mu_c(x,xy) + 1 \\ &= \Big[\prod_{p\in\mathbb{P}} \mu_c(x, xp^{(xy)(p)-x(p)})\Big] + 1 \\ &= \Big[\prod_{p\in\mathbb{P}} \mu_c(x, xp^{y(p)})\Big] + 1 \\ &= (-1)^k + 1, \end{aligned}$$

where $k \in \mathbb{N}$ is the number of distinct prime factors of y. Since $y = xy/x$, it follows that $\omega(x, xy) = k$, and therefore

$$\zeta(x,x)\mu_c(x,xy) + \zeta(x,xy)\mu_c(x,x) = (-1)^{\omega(x,xy)} + 1.$$

Since $(\zeta *_C \mu_c)(x, xy) = (-1)^{\omega(x,xy)} + 1$ if and only if $\omega(x, xy)$ is odd, it follows by Definition 7.2 that $\zeta *_C \mu_c$ is not C-discriminative, whereas by Definition 7.5 it is partially C-discriminative. □

Remark. Also $(-\zeta) *_C \mu_c$ and $\zeta *_C (-\mu_c)$ are partially C-discriminative.

The following theorems present necessary and sufficient conditions for a partially semifactorable function to be semifactorable.

Theorem 7.21. *Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be partially semifactorable. Then f is semifactorable if and only if*

$$\exists\, g, h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq] : f(g *_C h) = (fg) *_C (fh),$$

where $g *_C h$ is partially C-discriminative.

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. Let us first assume that f is semifactorable. Then by Theorem 7.15 $f(\zeta *_C \mu_c) = (f\zeta) *_C (f\mu_c)$, where (by Theorem 7.20) $\zeta *_C \mu_c$ is partially C-discriminative.

Let us next assume that

$$\exists\, g, h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq] : f(g *_C h) = (fg) *_C (fh),$$

where $g *_C h$ is partially C-discriminative. (i) By the partial semifactorability of f

$$\forall\, x \in \mathbb{Z}_+ : f(x,x) = 1.$$

(ii) Let $x, y \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$, where $x = \prod_{p \in \mathbb{P}} p^{x(p)}$ and $y = \prod_{p \in \mathbb{P}} p^{y(p)}$. Let us use induction on $\omega(x, y) \in \mathbb{N}$, where $\omega(x, y)$ stands for the number of distinct prime factors of y/x (see Definition 5.6), to show that

$$f(x,y) = \prod_{p \in \mathbb{P}} f(x, xp^{y(p)-x(p)}).$$

If $\omega(x, y) = 0$, then $x(p) = y(p)$ for all $p \in \mathbb{P}$, and therefore by the partial semifactorability of f

$$f(x,y) = f(x,x) = 1 = \prod_{p \in \mathbb{P}} f(x, xp^{y(p)-x(p)}).$$

Let $\omega(x, y) = n > 0$, and let us assume that

$$\forall\, u \in \mathbb{Z}_+ : (x \trianglelefteq u \trianglelefteq y \text{ and } \omega(x,u) < n) \Rightarrow f(x,u) = \prod_{p \in \mathbb{P}} f(x, xp^{u(p)-x(p)}),$$

where $u = \prod_{p\in\mathbb{P}} p^{u(p)}$. Let us assume that $g, h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ satisfy the assumption, i.e. $g *_C h$ is partially C-discriminative and $f(g *_C h) = (fg) *_C (fh)$. Then by Lemmas 5.15 and 5.3, the partial semifactorability of f, Lemma 7.3, the induction hypothesis, and Lemma 7.2 it follows (as in the proof of Theorem 7.17, the only difference being that (i) is replaced by the partial semifactorability of f) that

$$f(x,y) \sum_{\substack{\mathrm{gcf}(z,w)=x\\ \mathrm{lcm}(z,w)=y\\ z\neq x,\, w\neq y\\ z\neq y,\, w\neq x}} g(x,z)h(x,w) = \prod_{p\in\mathbb{P}} f(x, xp^{y(p)-x(p)}) \sum_{\substack{\mathrm{gcf}(z,w)=x\\ \mathrm{lcm}(z,w)=y\\ z\neq x,\, w\neq y\\ z\neq y,\, w\neq x}} g(x,z)h(x,w).$$

If $\omega(x,y) = n$ is odd, then by the partial semifactorability of f

$$f(x,y) = \prod_{p\in\mathbb{P}} f(x, xp^{y(p)-x(p)}).$$

Let us assume that $\omega(x,y) = n$ is even. Since $\omega(x,y) = n > 0$, it follows that $y = xu$, where $u \in \mathbb{Z}_+$ is such that $u \neq 1$, and therefore, since $g *_C h$ is partially C-discriminative, it follows that

$$(g *_C h)(x,y) \neq g(x,x)h(x,y) + g(x,y)h(x,x).$$

On the other hand (by Lemmas 5.15 and 5.3)

$$(g *_C h)(x,y) = g(x,x)h(x,y) + g(x,y)h(x,x) + \sum_{\substack{\mathrm{gcf}(z,w)=x\\ \mathrm{lcm}(z,w)=y\\ z\neq x,\, w\neq y\\ z\neq y,\, w\neq x}} g(x,z)h(x,w),$$

and therefore

$$\sum_{\substack{\mathrm{gcf}(z,w)=x\\ \mathrm{lcm}(z,w)=y\\ z\neq x,\, w\neq y\\ z\neq y,\, w\neq x}} g(x,z)h(x,w) \neq 0.$$

Thus

$$f(x,y) = \prod_{p\in\mathbb{P}} f(x, xp^{y(p)-x(p)}),$$

and therefore

$$\forall\, x, y \in \mathbb{Z}_+ : x \trianglelefteq y \Rightarrow f(x,y) = \prod_{p\in\mathbb{P}} f(x, xp^{y(p)-x(p)}),$$

where $x = \prod_{p\in\mathbb{P}} p^{x(p)}$ and $y = \prod_{p\in\mathbb{P}} p^{y(p)}$. Thus by Theorem 7.5 f is semifactorable. $\square$

Lemma 7.4. *Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. Then $f\delta = \delta$ if and only if*

$$\forall\, x \in \mathbb{Z}_+ : f(x,x) = 1.$$

Proof. Elementary. $\square$

Theorem 7.22. *Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be partially semifactorable. Then f is semifactorable if and only if*

$$f^{*_C -1} = f\mu_c.$$

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be partially semifactorable. Let us first assume that f is semifactorable. Then by Theorem 7.15 $f(\zeta *_C \mu_c) = (f\zeta) *_C (f\mu_c)$, and therefore by Theorem 5.9 and Lemma 7.4 $f *_C (f\mu_c) = \delta$. Since $f *_C (f\mu_c) = \delta$, it follows by Theorem 5.35 that $(f\mu_c) *_C f = \delta$. Thus $f\mu_c$ is the C-convolution inverse of f, i.e. $f^{*_C -1} = f\mu_c$.

Let us next assume that $f^{*_C -1} = f\mu_c$. Since by Theorem 7.20 $\zeta *_C \mu_c$ is partially C-discriminative and from $f *_C (f\mu_c) = \delta$ it follows by Theorem 5.9 and Lemma 7.4 that $f(\zeta *_C \mu_c) = (f\zeta) *_C (f\mu_c)$, it follows by Theorem 7.21 that f is semifactorable. $\square$

Theorem 7.23. *Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be partially semifactorable. Then f is semifactorable if and only if*

$$\forall\, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq] : \Big[\forall\, x \in \mathbb{Z}_+ : g(x,x) \neq 0 \Rightarrow (fg)^{*_C -1} = fg^{*_C -1}\Big].$$

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be partially semifactorable. Let us first assume that f is semifactorable. Let $g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be such that

$$\forall\, x \in \mathbb{Z}_+ : g(x,x) \neq 0.$$

Then by Theorem 5.38 g has a C-convolution inverse, and therefore by Theorems 7.15 and 5.36

$$(fg) *_C (fg^{*_C -1}) = f(g *_C g^{*_C -1}) = f\delta.$$

From the semifactorability of f it follows by Lemma 7.4 that $f\delta = \delta$, and therefore $(fg) *_C (fg^{*_C -1}) = \delta$. Since $(fg) *_C (fg^{*_C -1}) = \delta$, it follows by Theorem 5.35 that $(fg^{*_C -1}) *_C (fg) = \delta$. Thus $fg^{*_C -1}$ is the C-convolution inverse of fg, i.e. $(fg)^{*_C -1} = fg^{*_C -1}$.

Let us next assume that

$$\forall\, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq] : \Big[\forall\, x \in \mathbb{Z}_+ : g(x,x) \neq 0 \Rightarrow (fg)^{*_C -1} = fg^{*_C -1}\Big].$$

Since

$$\forall\, x \in \mathbb{Z}_+ : \zeta(x,x) \neq 0,$$

it follows by Theorems 5.38 and 5.9, and the assumption that

$$f^{*_C -1} = (f\zeta)^{*_C -1} = f\zeta^{*_C -1} = f\mu_c.$$

Thus by Theorem 7.22 f is semifactorable. $\square$

Remark. Replacing the phrase '$\omega(x, y)$ is odd' in Definitions 7.3 and 7.4 with the phrase '$\omega(x, y) = 1$ or $\omega(x, y) > 1$ and even' and the phrase '$\omega(x, xy)$ is odd' in Definition 7.5 with the phrase '$\omega(x, xy) = 1$ or $\omega(x, xy) > 1$ and even' results to a corresponding set of definitions with 'even omega'. This, in turn, suggests an alternative way, parallel to Theorems 7.20, 7.21, 7.22, and 7.23, to give necessary and sufficient conditions for an 'even-omega-semifactorable' function to be semifactorable. However, this parallel approach depends on finding an 'even-omega-C-discriminative' C-convolution, and more importantly, on the existence of such C-convolution. The present study leaves this particular problem unanswered.

It is evident that the semifactorability and the translation invariance of an arithmetic incidence function are properties that do not depend on each other. Of these two properties, the translation invariance presents more systematic and widespread conditions for a function and its values to fulfill compared to the semifactorability which, in turn, can be characterized, in a sense, as a more local property. In light of this observation, a natural approach is to assume the translation invariance of a function and, using this assumption, study its semifactorability.

Let us next investigate briefly translation invariant arithmetic incidence functions and their semifactorability. The following theorem presents a necessary and sufficient condition for a translation invariant function to be semifactorable.

Theorem 7.24. *Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be translation invariant. Then f is semifactorable if and only if*

(i) $\forall\, x \in \mathbb{Z}_+ : f(x, x) = 1,$

(ii) $\forall\, x, y, z, w \in \mathbb{Z}_+ : \Big[\mathrm{gcf}(z, w) = x \text{ and } \mathrm{lcm}(z, w) = y\Big]$
$\qquad\qquad \Rightarrow f(x, y) = f(x, z)f(z, y).$

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be translation invariant. If $x, y, z, w \in \mathbb{Z}_+$ are such that $\mathrm{gcf}(z, w) = x$ and $\mathrm{lcm}(z, w) = y$, then by Theorem 6.5

$$f(x, w) = f(\mathrm{gcf}(w, z), w) = f(z, \mathrm{lcm}(w, z)) = f(z, y).$$

Thus by Theorem 7.2 f is semifactorable if and only if (i) and (ii) hold. $\square$

Remark. Both the translation invariance and the semifactorability have an important role concerning the subsequent study concerning the factorability of an arithmetic incidence function, and Theorem 7.24 plays also a minor part in this study.

7.2 Semicompressibility

The duality of the concepts of the greatest common factor and the least common multiple suggests also a property that is analogous to the semifactorability of an arithmetic incidence function.

Definition 7.6. A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is *semicompressible* if

(i) $\forall\, x \in \mathbb{Z}_+ : f(x,x) = 1$,

(ii) $\forall\, x,y,z \in \mathbb{Z}_+ : y = \operatorname{lcm}(x,z) \Rightarrow f(\operatorname{gcf}(x,z),y) = f(x,y)f(z,y)$.

Remark. The use of the term 'compressible' is motivated by the way how the function values at specific arguments satisfying the required conditions can be multiplied together, in other words compressed, in order to produce a function value at arguments that are, in effect, common factors of the original arguments.

The following theorem presents a characterization of a semicompressible function, and it is essentially a reformulation of the definition of a semicompressible function.

Theorem 7.25. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is semicompressible if and only if*

(i) $\forall\, x \in \mathbb{Z}_+ : f(x,x) = 1$,

(ii) $\forall\, x,y,z,w \in \mathbb{Z}_+ : \Big[\operatorname{gcf}(z,w) = x \textit{ and } \operatorname{lcm}(z,w) = y\Big]$
$\Rightarrow f(x,y) = f(z,y)f(w,y)$.

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. Let us first assume that f is semicompressible. (i) By the semicompressibility of f

$$\forall\, x \in \mathbb{Z}_+ : f(x,x) = 1.$$

(ii) Let $x,y,z,w \in \mathbb{Z}_+$ be such that $\operatorname{gcf}(z,w) = x$ and $\operatorname{lcm}(z,w) = y$. Then by the semicompressibility of f

$$f(x,y) = f(\operatorname{gcf}(z,w),y) = f(z,y)f(w,y).$$

Thus (i) and (ii) hold.

Let us next assume that (i) and (ii) hold. Let $x,y,z \in \mathbb{Z}_+$ be such that $y = \operatorname{lcm}(x,z)$. Then by (ii) $f(\operatorname{gcf}(x,z),y) = f(x,y)f(z,y)$. Thus f is semicompressible. $\square$

Lemma 7.5. *If $x,y,z,w \in \mathbb{Z}_+$ are such that $x \trianglelefteq y$, $z \trianglelefteq w$, and $\operatorname{lcm}(x,z) = \operatorname{lcm}(y,w)$, then*

$$\forall\, p \in \mathbb{P} : \Big[\min\{x(p),z(p)\} = x(p),\ \min\{y(p),w(p)\} = y(p),\ \textit{and } z(p) = w(p)\Big]$$
$$\textit{or } \Big[\min\{x(p),z(p)\} = z(p),\ \min\{y(p),w(p)\} = w(p),\ \textit{and } x(p) = y(p)\Big],$$

where $x = \prod_{p\in\mathbb{P}} p^{x(p)}$, $y = \prod_{p\in\mathbb{P}} p^{y(p)}$, $z = \prod_{p\in\mathbb{P}} p^{z(p)}$, and $w = \prod_{p\in\mathbb{P}} p^{w(p)}$.

Proof. Let $x, y, z, w \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$, $z \trianglelefteq w$, and $\operatorname{lcm}(x, z) = \operatorname{lcm}(y, w)$. Since $\operatorname{lcm}(x, z) = \operatorname{lcm}(y, w)$, it follows that

$$\forall\, p \in \mathbb{P} : \max\{x(p), z(p)\} = \max\{y(p), w(p)\},$$

where $x = \prod_{p\in\mathbb{P}} p^{x(p)}$, $y = \prod_{p\in\mathbb{P}} p^{y(p)}$, $z = \prod_{p\in\mathbb{P}} p^{z(p)}$, and $w = \prod_{p\in\mathbb{P}} p^{w(p)}$. Let $p \in \mathbb{P}$. Then (i) $\min\{x(p), z(p)\} = x(p)$ or (ii) $\min\{x(p), z(p)\} = z(p)$.

(i) Let $\min\{x(p), z(p)\} = x(p)$. Then $\max\{x(p), z(p)\} = z(p)$, and therefore also $\max\{y(p), w(p)\} = z(p)$. Thus $y(p) \leq z(p)$ and $w(p) \leq z(p)$. Since $z \trianglelefteq w$, it follows that $z(p) \leq w(p)$, and therefore $z(p) = w(p)$. Since $y(p) \leq z(p)$ and $z(p) \leq w(p)$, it follows that $y(p) \leq w(p)$, and therefore $\min\{y(p), w(p)\} = y(p)$.

(ii) Let $\min\{x(p), z(p)\} = z(p)$. Then $\max\{x(p), z(p)\} = x(p)$, and therefore also $\max\{y(p), w(p)\} = x(p)$. Thus $y(p) \leq x(p)$ and $w(p) \leq x(p)$. Since $x \trianglelefteq y$, it follows that $x(p) \leq y(p)$, and therefore $x(p) = y(p)$. Since $w(p) \leq x(p)$ and $x(p) \leq y(p)$, it follows that $w(p) \leq y(p)$, and therefore $\min\{y(p), w(p)\} = w(p)$. $\square$

The following theorem presents a prime related characterization of a semicompressible function, and therefore, in other words, it gives a necessary and sufficient condition for a function to be semicompressible.

Theorem 7.26. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is semicompressible if and only if*

(i) $\forall\, x \in \mathbb{Z}_+ : f(x, x) = 1$,

(ii) $\forall\, x, y \in \mathbb{Z}_+ : x \trianglelefteq y \Rightarrow f(x, y) = \prod_{p\in\mathbb{P}} f(y/p^{y(p)-x(p)}, y)$,
where $x = \prod_{p\in\mathbb{P}} p^{x(p)}$ *and* $y = \prod_{p\in\mathbb{P}} p^{y(p)}$.

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. Let us first assume that f is semicompressible. (i) By the semicompressibility of f

$$\forall\, x \in \mathbb{Z}_+ : f(x, x) = 1.$$

(ii) Let $x, y \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$, where $x = \prod_{p\in\mathbb{P}} p^{x(p)}$ and $y = \prod_{p\in\mathbb{P}} p^{y(p)}$. Let us assume that $y = 1$. Then $x = 1$. Thus $x(p) = 0$ and $y(p) = 0$ for all $p \in \mathbb{P}$, and therefore by (i)

$$f(x, y) = f(1, 1) = \prod_{p\in\mathbb{P}} f(1, 1) = \prod_{p\in\mathbb{P}} f(y/p^{y(p)-x(p)}, y).$$

Let us assume that $y \neq 1$. Then

$$x = \prod_{p\in\mathbb{P}} p^{x(p)} = \prod_{i=1}^{n} p_i^{x(p_i)} \quad \text{and} \quad y = \prod_{p\in\mathbb{P}} p^{y(p)} = \prod_{i=1}^{n} p_i^{y(p_i)},$$

where $n \in \mathbb{Z}_+$ stands for the number of distinct primes in the prime factorization of the element y and $x(p_i) = 0$, $1 \le i \le n$, if $p_i \in \mathbb{P}$ is not a prime factor of the element x. Let us note that if $z \in \mathbb{Z}_+$ is such that $xz = y$, then

$$\forall\, p \in \mathbb{P} : x(p) + z(p) = y(p),$$

and therefore for all $n \in \mathbb{Z}_+$

$$x = y/z = y/\prod_{i=1}^{n} p_i^{z(p_i)} = y/\prod_{i=1}^{n} p_i^{y(p_i)-x(p_i)},$$

where $z(p_i) = 0$, $1 \le i \le n$, if $p_i \in \mathbb{P}$ is not a prime factor of the element z. Let us use induction on $n \in \mathbb{Z}_+$ to show that

$$\forall\, n \in \mathbb{Z}_+ : f(x, y) = \prod_{i=1}^{n} f(y/p_i^{y(p_i)-x(p_i)}, y)$$

from which the result follows. If $n = 1$, then

$$f(x, y) = f(y/\prod_{i=1}^{1} p_i^{y(p_i)-x(p_i)}, y) = f(y/p^{y(p_1)-x(p_1)}, y) = \prod_{i=1}^{1} f(y/p_i^{y(p_i)-x(p_i)}, y).$$

Let $n \in \mathbb{Z}_+$, and let us assume that the claim holds for n, that is,

$$f(y/\prod_{i=1}^{n} p_i^{y(p_i)-x(p_i)}, y) = \prod_{i=1}^{n} f(y/p_i^{y(p_i)-x(p_i)}, y).$$

Let the number of distinct primes in the prime factorization of the element y be $n + 1$. Then

$$x = \prod_{i=1}^{n+1} p_i^{x(p_i)} \quad \text{and} \quad y = \prod_{i=1}^{n+1} p_i^{y(p_i)}.$$

Since

$$\operatorname{gcf}(xp_{n+1}^{y(p_{n+1})-x(p_{n+1})}, x\prod_{i=1}^{n} p_i^{y(p_i)-x(p_i)}) = x$$

and

$$\operatorname{lcm}(xp_{n+1}^{y(p_{n+1})-x(p_{n+1})}, x\prod_{i=1}^{n} p_i^{y(p_i)-x(p_i)}) = x\prod_{i=1}^{n+1} p_i^{y(p_i)-x(p_i)} = y,$$

it follows that

$$\operatorname{gcf}(y/\Big[\prod_{i=1}^{n} p_i^{y(p_i)-x(p_i)}\Big], y/p_{n+1}^{y(p_{n+1})-x(p_{n+1})}) = x$$

and

$$\operatorname{lcm}(y/\Big[\prod_{i=1}^{n} p_i^{y(p_i)-x(p_i)}\Big], y/p_{n+1}^{y(p_{n+1})-x(p_{n+1})}) = y,$$

and therefore by the semicompressibility of f and the induction hypothesis

$$\begin{aligned}
f(x,y) &= f(\mathrm{gcf}(y/\prod_{i=1}^{n} p_i^{y(p_i)-x(p_i)}, y/p_{n+1}^{y(p_{n+1})-x(p_{n+1})}), y) \\
&= f(y/\prod_{i=1}^{n} p_i^{y(p_i)-x(p_i)}, y) f(y/p_{n+1}^{y(p_{n+1})-x(p_{n+i})}, y) \\
&= \Big[\prod_{i=1}^{n} f(y/p_i^{y(p_i)-x(p_i)}, y)\Big] f(y/p_{n+1}^{y(p_{n+1})-x(p_{n+i})}, y) \\
&= \prod_{i=1}^{n+1} f(y/p_i^{y(p_i)-x(p_i)}, y).
\end{aligned}$$

Thus

$$\forall\, n \in \mathbb{Z}_+ : f(x,y) = \prod_{i=1}^{n} f(y/p_i^{y(p_i)-x(p_i)}, y).$$

From the semicompressibility of f it follows that

$$\forall\, p \in \mathbb{P} : x(p) = y(p) = 0 \Rightarrow f(y/p^{y(p)-x(p)}, y) = 1,$$

and therefore

$$f(x,y) = \prod_{p\in\mathbb{P}} f(y/p^{y(p)-x(p)}).$$

Thus (i) and (ii) hold.

Let us next assume that (i) and (ii) hold. Let $x, y, z \in \mathbb{Z}_+$ be such that $y = \mathrm{lcm}(x,z)$. Let us note that since $x \trianglelefteq y$ and $z \trianglelefteq y$, it follows by Lemma 7.5 that

$$\begin{aligned}
\forall\, p \in \mathbb{P} :& \Big[\min\{x(p), z(p)\} = x(p) \text{ and } z(p) = y(p)\Big] \\
\text{or } & \Big[\min\{x(p), z(p)\} = z(p) \text{ and } x(p) = y(p)\Big],
\end{aligned}$$

where $x = \prod_{p\in\mathbb{P}} p^{x(p)}$, $y = \prod_{p\in\mathbb{P}} p^{y(p)}$, and $z = \prod_{p\in\mathbb{P}} p^{z(p)}$. Let us specifically note that if $\min\{x(p), z(p)\} = x(p)$ and $z(p) = y(p)$, then by (i)

$$\begin{aligned}
f(y/p^{y(p)-\min\{x(p),z(p)\}}, y) &= f(y/p^{y(p)-x(p)}, y) \\
&= f(y/p^{y(p)-x(p)}, y) f(y,y) \\
&= f(y/p^{y(p)-x(p)}, y) f(y/p^{y(p)-z(p)}, y).
\end{aligned}$$

Correspondingly, if $\min\{x(p), z(p)\} = z(p)$ and $x(p) = y(p)$, then by (i)

$$\begin{aligned}
f(y/p^{y(p)-\min\{x(p),z(p)\}}, y) &= f(y/p^{y(p)-z(p)}, y) \\
&= f(y,y) f(y/p^{y(p)-z(p)}, y) \\
&= f(y/p^{y(p)-x(p)}, y) f(y/p^{y(p)-z(p)}, y).
\end{aligned}$$

Thus

$$\forall\, p \in \mathbb{P} : f(y/p^{y(p)-\min\{x(p),z(p)\}}, y) = f(y/p^{y(p)-x(p)}, y) f(y/p^{y(p)-z(p)}, y),$$

and therefore

$$\forall\, p \in \mathbb{P} : f(y/p^{y(p)-\mathrm{gcf}(x,z)(p)}, y) = f(y/p^{y(p)-x(p)}, y) f(y/p^{y(p)-z(p)}, y).$$

Thus by (ii)

$$\begin{aligned} f(\mathrm{gcf}(x,z), y) &= \prod_{p\in\mathbb{P}} f(y/p^{y(p)-\mathrm{gcf}(x,z)(p)}, y) \\ &= \prod_{p\in\mathbb{P}} f(y/p^{y(p)-x(p)}, y) f(y/p^{y(p)-z(p)}, y) \\ &= \Big[\prod_{p\in\mathbb{P}} f(y/p^{y(p)-x(p)}, y)\Big]\Big[\prod_{p\in\mathbb{P}} f(y/p^{y(p)-z(p)}, y)\Big] \\ &= f(x,y) f(z,y). \end{aligned}$$

Thus f is semicompressible. $\square$

The following theorem presents another characterization of a semicompressible function, and it is closely related to Theorem 7.26.

Theorem 7.27. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is semicompressible if and only if*

(i) $\forall\, x \in \mathbb{Z}_+ : f(x,x) = 1$,

(ii) $\forall\, x, y, z \in \mathbb{Z}_+ : \Big[x \trianglelefteq y \text{ and } z \trianglelefteq y\Big]$
$\Rightarrow f(\mathrm{gcf}(x,z), y) f(\mathrm{lcm}(x,z), y) = f(x,y) f(z,y).$

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. Let us first assume that f is semicompressible. (i) By the semicompressibility of f

$$\forall\, x \in \mathbb{Z}_+ : f(x,x) = 1.$$

(ii) Let $x, y, z \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$ and $z \trianglelefteq y$. Let us note that

$$\begin{aligned} \forall\, p \in \mathbb{P} : &\Big[\min\{x(p), z(p)\} = x(p) \text{ and } \max\{x(p), z(p)\} = z(p)\Big] \\ \text{or } &\Big[\min\{x(p), z(p)\} = z(p) \text{ and } \max\{x(p), z(p)\} = x(p)\Big], \end{aligned}$$

where $y = \prod_{p\in\mathbb{P}} p^{y(p)}$ and $z = \prod_{p\in\mathbb{P}} p^{z(p)}$. Let us specifically note that if $\min\{x(p), z(p)\} = x(p)$ and $\max\{x(p), z(p)\} = z(p)$, then

$$\begin{aligned} &f(y/p^{y(p)-\min\{x(p),z(p)\}}, y) f(y/p^{y(p)-\max\{x(p),z(p)\}}, y) \\ &\quad = f(y/p^{y(p)-x(p)}, y) f(y/p^{y(p)-z(p)}, y). \end{aligned}$$

Correspondingly, if $\min\{x(p), z(p)\} = z(p)$ and $\max\{x(p), z(p)\} = x(p)$, then

$$\begin{aligned} &f(y/p^{y(p)-\min\{x(p),z(p)\}}, y)f(y/p^{y(p)-\max\{x(p),z(p)\}}, y) \\ &\quad = f(y/p^{y(p)-z(p)}, y)f(y/p^{y(p)-x(p)}, y). \end{aligned}$$

Thus

$$\begin{aligned} \forall\, p \in \mathbb{P} : &f(y/p^{y(p)-\min\{x(p),z(p)\}}, y)f(y/p^{y(p)-\max\{x(p),z(p)\}}, y) \\ &= f(y/p^{y(p)-x(p)}, y)f(y/p^{y(p)-z(p)}, y), \end{aligned}$$

and therefore

$$\begin{aligned} \forall\, p \in \mathbb{P} : &f(y/p^{y(p)-\operatorname{gcf}(x,z)(p)}, y)f(y/p^{y(p)-\operatorname{lcm}(x,z)(p)}, y) \\ &= f(y/p^{y(p)-x(p)}, y)f(y/p^{y(p)-z(p)}, y). \end{aligned}$$

Thus by Theorem 7.26

$$\begin{aligned} &f(\operatorname{gcf}(x,z), y)f(\operatorname{lcm}(x,z), y) \\ &\quad = \Big[\prod_{p\in\mathbb{P}} f(y/p^{y(p)-\operatorname{gcf}(x,z)(p)}, y)\Big]\Big[\prod_{p\in\mathbb{P}} f(y/p^{y(p)-\operatorname{lcm}(x,z)(p)}, y)\Big] \\ &\quad = \prod_{p\in\mathbb{P}} f(y/p^{y(p)-\operatorname{gcf}(x,z)(p)}, y)f(y/p^{y(p)-\operatorname{lcm}(x,z)(p)}, y) \\ &\quad = \prod_{p\in\mathbb{P}} f(y/p^{y(p)-x(p)}, y)f(y/p^{y(p)-z(p)}, y) \\ &\quad = \Big[\prod_{p\in\mathbb{P}} f(y/p^{y(p)-x(p)}, y)\Big]\Big[\prod_{p\in\mathbb{P}} f(y/p^{y(p)-z(p)}, y)\Big] \\ &\quad = f(x, y)f(z, y). \end{aligned}$$

Thus (i) and (ii) hold.

Let us next assume that (i) and (ii) hold. Let $x, y, z \in \mathbb{Z}_+$ be such that $y = \operatorname{lcm}(x, z)$. Then $x \trianglelefteq y$ and $z \trianglelefteq y$, and therefore by (i) and (ii)

$$f(\operatorname{gcf}(x,z), y) = f(\operatorname{gcf}(x,z), y)f(\operatorname{lcm}(x,z), y) = f(x, y)f(z, y).$$

Thus f is semicompressible. $\square$

Theorem 7.28. *Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be translation invariant. Then f is semicompressible if and only if it is semifactorable.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be translation invariant. Let us first assume that f is semicompressible. Let $x, y, w \in \mathbb{Z}_+$ be such that $x = \operatorname{gcf}(y, w)$. Then by the semicompressibility of f and Theorem 6.5

$$\begin{aligned} f(x, \operatorname{lcm}(y, w)) &= f(\operatorname{gcf}(y, w), \operatorname{lcm}(y, w)) \\ &= f(y, \operatorname{lcm}(y, w))f(w, \operatorname{lcm}(y, w)) \\ &= f(y, \operatorname{lcm}(w, y))f(w, \operatorname{lcm}(y, w)) \\ &= f(\operatorname{gcf}(w, y), w)f(\operatorname{gcf}(y, w), y) \\ &= f(x, w)f(x, y) \\ &= f(x, y)f(x, w). \end{aligned}$$

Thus by Definition 7.1 f is semifactorable.

Let us next assume that f is semifactorable. Let $x, y, z \in \mathbb{Z}_+$ be such that $y = \operatorname{lcm}(x, z)$. Then by the semifactorability of f and Theorem 6.5

$$\begin{aligned}
f(\operatorname{gcf}(x,z), y) &= f(\operatorname{gcf}(x,z), \operatorname{lcm}(x,z)) \\
&= f(\operatorname{gcf}(x,z), x) f(\operatorname{gcf}(x,z), z) \\
&= f(\operatorname{gcf}(x,z), x) f(\operatorname{gcf}(z,x), z) \\
&= f(z, \operatorname{lcm}(x,z)) f(x, \operatorname{lcm}(z,x)) \\
&= f(z,y) f(x,y) \\
&= f(x,y) f(z,y).
\end{aligned}$$

Thus f is semicompressible. □

Let us next demonstrate the usefulness of Theorem 7.28. Since by Theorems 6.1, 6.6, 6.10, and 6.14 the functions $\zeta, \delta, \mu, \mu_c \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ are translation invariant and by Theorems 7.1, 7.3, 7.9, and 7.13 they are semifactorable, it follows by Theorem 7.28 that they are also semicompressible.

Let us next demonstrate that the semifactorability and the semicompressibility of a function are properties that do not depend on each other. Let $p_1, p_2 \in \mathbb{P}$ be distinct, and let us define the functions $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ as follows:

$$f : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C} : f(x,y) = \begin{cases} 1 & \text{if } x = y, \\ 1 & \text{if } x = 1 \text{ and } y = p_1, \\ 1 & \text{if } x = 1 \text{ and } y = p_2, \\ 1 & \text{if } x = 1 \text{ and } y = p_1 p_2, \\ 0 & \text{otherwise.} \end{cases}$$

$$g : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C} : g(x,y) = \begin{cases} 1 & \text{if } x = y, \\ 1 & \text{if } x = 1 \text{ and } y = p_1 p_2, \\ 1 & \text{if } x = p_1 \text{ and } y = p_1 p_2, \\ 1 & \text{if } x = p_2 \text{ and } y = p_1 p_2, \\ 0 & \text{otherwise.} \end{cases}$$

Then, by Theorems 7.2 and 7.25, f is semifactorable but not semicompressible, whereas g is semicompressible but not semifactorable. This establishes that if a function is semifactorable, then it is not necessarily semicompressible. Correspondingly, if a function is semicompressible, then it is not necessarily semifactorable.

The definitions of the D-convolution and the semifactorability of arithmetic incidence functions are, by choice, in a sense in line with each other, and therefore the D-convolution preserves the property of semifactorability and the D-convolution inverses of semifactorable functions are, without any additional requirements, also semifactorable. In contrast to this, the same does not hold in general for the semicompressibility. In order to verify this, let $p_1, p_2 \in \mathbb{P}$ be distinct, and let us define the functions $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ as follows:

$$f : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C} : f(x,y) = \begin{cases} 1 & \text{if } x = y, \\ 1 & \text{if } x = p_1 \text{ and } y = p_1p_2, \\ 0 & \text{otherwise.} \end{cases}$$

$$g : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C} : g(x,y) = \begin{cases} 1 & \text{if } x = y, \\ 1 & \text{if } x = p_2 \text{ and } y = p_1p_2, \\ 0 & \text{otherwise.} \end{cases}$$

Then by Theorem 7.26 f and g are semicompressible and

$$\begin{aligned}(f * g)(\mathrm{gcf}(p_1, p_2), p_1p_2) &= (f * g)(1, p_1p_2) \\ &= \sum_{\substack{1 \trianglelefteq z \trianglelefteq p_1p_2 \\ zw = 1 \cdot p_1p_2}} f(1, z)g(1, w) \\ &= f(1, 1)g(1, p_1p_2) + f(1, p_1)g(1, p_2) \\ &\quad + f(1, p_2)g(1, p_1) + f(1, p_1p_2)g(1, 1) \\ &= 0,\end{aligned}$$

$$\begin{aligned}(f * g)(p_1, p_1p_2) &= \sum_{\substack{p_1 \trianglelefteq z \trianglelefteq p_1p_2 \\ zw = p_1 \cdot p_1p_2}} f(p_1, z)g(p_1, w) \\ &= f(p_1, p_1)g(p_1, p_1p_2) + f(p_1, p_1p_2)g(p_1, p_1) \\ &= 1,\end{aligned}$$

and

$$\begin{aligned}(f * g)(p_2, p_1p_2) &= \sum_{\substack{p_2 \trianglelefteq z \trianglelefteq p_1p_2 \\ zw = p_2 \cdot p_1p_2}} f(p_2, z)g(p_2, w) \\ &= f(p_2, p_2)g(p_2, p_1p_2) + f(p_2, p_1p_2)g(p_2, p_2) \\ &= 1.\end{aligned}$$

Thus $p_1p_2 = \mathrm{lcm}(p_1, p_2)$ but

$$(f * g)(\mathrm{gcf}(p_1, p_2), p_1p_2) \neq (f * g)(p_1, p_1p_2)(f * g)(p_2, p_1p_2),$$

and therefore $f * g$ is not semicompressible. This establishes that the D-convolution does not necessarily preserve the semicompressibility. Let us define the function $h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ as follows:

$$h : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C} : h(x,y) = \begin{cases} 1 & \text{if } x = y, \\ 1 & \text{if } x = 1 \text{ and } y = p_1p_2, \\ 1 & \text{if } x = p_1 \text{ and } y = p_1p_2, \\ 1 & \text{if } x = p_2 \text{ and } y = p_1p_2, \\ 0 & \text{otherwise.} \end{cases}$$

Then by Theorem 7.26 h is semicompressible and by Theorem 5.21

$$\begin{aligned} h^{*-1}(\operatorname{gcf}(p_1, p_2), p_1p_2) &= h^{*-1}(1, p_1p_2) \\ &= -\sum_{\substack{1 \trianglelefteq z \triangleleft p_1p_2 \\ zw = 1 \cdot p_1p_2}} h^{*-1}(1, z)h(1, w) \\ &= -\Big[h^{*-1}(1, 1)h(1, p_1p_2) \\ &\qquad + h^{*-1}(1, p_1)h(1, p_2) + h^{*-1}(1, p_2)h(1, p_1)\Big] \\ &= -1, \end{aligned}$$

$$\begin{aligned} h^{*-1}(p_1, p_1p_2) &= -\sum_{\substack{p_1 \trianglelefteq z \triangleleft p_1p_2 \\ zw = p_1 \cdot p_1p_2}} h^{*-1}(p_1, z)h(p_1, w) \\ &= -h^{*-1}(p_1, p_1)h(p_1, p_1p_2) \\ &= -1, \end{aligned}$$

and

$$\begin{aligned} h^{*-1}(p_2, p_1p_2) &= -\sum_{\substack{p_2 \trianglelefteq z \triangleleft p_1p_2 \\ zw = p_2 \cdot p_1p_2}} h^{*-1}(p_2, z)h(p_2, w) \\ &= -h^{*-1}(p_2, p_2)h(p_2, p_1p_2) \\ &= -1. \end{aligned}$$

Thus $p_1p_2 = \operatorname{lcm}(p_1, p_2)$ but

$$h^{*-1}(\operatorname{gcf}(p_1, p_2), p_1p_2) \neq h^{*-1}(p_1, p_1p_2)h^{*-1}(p_2, p_1p_2),$$

and therefore h^{*-1} is not semicompressible. Thus $h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is an example of a semicompressible function associated with a D-convolution inverse that is not semicompressible.

However, by adding the requirement of translation invariance it is guaranteed that the D-convolution preserves the semicompressibility and that the D-convolution inverse of a semicompressible function is also semicompressible.

Theorem 7.29. *If $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ are translation invariant and semicompressible, then $f * g$ is semicompressible.*

Proof. Let $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be translation invariant and semicompressible. Then by Theorem 7.28 both f and g are semifactorable, and therefore by Theorem 7.7 $f * g$ is semifactorable. Since by Theorem 6.8 also $f * g$ is translation invariant, it follows by Theorem 7.28 that $f * g$ is semicompressible. □

Theorem 7.30. *If $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is translation invariant and semicompressible, then f^{*-1} is semicompressible.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be translation invariant and semicompressible. Then by Theorem 7.28 f is semifactorable, and therefore by Theorem 7.8 f^{*-1} is semifactorable. Since by Theorem 6.9 also f^{*-1} is translation invariant, it follows by Theorem 7.28 that f^{*-1} is semicompressible. □

As in the case of the D-convolution, it is also the case that the C-convolution does not necessarily preserve the semicompressibility and that the C-convolution inverse of a semicompressible function is not necessarily semicompressible. In order to verify this, let us recollect that if $x, y \in \mathbb{Z}_+$ are such that the interval $[\,x, y\,]$ is a Boolean lattice (see Definition 2.25), then the D-convolution and the C-convolution coincide for these x and y, i.e.

$$\forall\, f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq] : (f * g)(x, y) = (f *_C g)(x, y).$$

In light of this observation, the functions $f, g, h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ and the related arguments that were given above to establish that the D-convolution does not necessarily preserve the semicompressibility and that the D-convolution inverse of a semicompressible function is not necessarily semicompressible, establish the same also in the case of the C-convolution.

Correspondingly, by adding the requirement of translation invariance it is guaranteed that the C-convolution preserves the semicompressibility and that the C-convolution inverse of a semicompressible function is also semicompressible.

Theorem 7.31. *If $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ are translation invariant and semicompressible, then $f *_C g$ is semicompressible.*

Proof. Let $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be translation invariant and semicompressible. Then by Theorem 7.28 both f and g are semifactorable, and therefore by Theorem 7.11 $f *_C g$ is semifactorable. Since by Theorem 6.12 also $f *_C g$ is translation invariant, it follows by Theorem 7.28 that $f *_C g$ is semicompressible. □

Theorem 7.32. *If $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is translation invariant and semicompressible, then f^{*_C-1} is semicompressible.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be translation invariant and semicompressible. Then by Theorem 7.28 f is semifactorable, and therefore by Theorem 7.12 f^{*_C-1} is semifactorable. Since by Theorem 6.13 also f^{*_C-1} is translation invariant, it follows by Theorem 7.28 that f^{*_C-1} is semicompressible. □

7.3 Complete Semifactorability

The notion of semifactorability of an arithmetic incidence function, as demonstrated above, generalizes the notion of multiplicativity of an arithmetic function of one variable. Correspondingly, the notion of complete semifactorability of an arithmetic incidence function generalizes the notion of complete multiplicativity of an arithmetic function of one variable.

Definition 7.7. A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is *completely semifactorable* if

(i) $\forall\, x \in \mathbb{Z}_+ : f(x,x) = 1$,

(ii) $\forall\, x,y,w \in \mathbb{Z}_+ : \Big[x \trianglelefteq y \text{ and } x \trianglelefteq w\Big]$
$\Rightarrow f(x, yw/x) = f(x,y)f(x,w)$.

Lemma 7.6. *If $x,y,w \in \mathbb{Z}_+$ are such that $x \trianglelefteq y$ and $x \trianglelefteq w$, then $x \trianglelefteq yw/x$.*

Proof. Elementary. □

Theorem 7.33. *The zeta function $\zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely semifactorable.*

Proof. (i) Let $x \in \mathbb{Z}_+$. Since $x \trianglelefteq x$, it follows that $\zeta(x,x) = 1$. (ii) Let $x,y,w \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$ and $x \trianglelefteq w$. Then by Lemma 7.6

$$\zeta(x, yw/x) = 1 = \zeta(x,y)\zeta(x,w).$$

By (i) and (ii) $\zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely semifactorable. □

The following theorem establishes that if an arithmetic incidence function is completely semifactorable, then it is necessarily also semifactorable.

Theorem 7.34. *If $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely semifactorable, then it is semifactorable.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be completely semifactorable. (i) By complete the semifactorability of f

$$\forall\, x \in \mathbb{Z}_+ : f(x,x) = 1.$$

(ii) Let $x,y,w \in \mathbb{Z}_+$ be such that $x = \gcd(y,w)$. Then $x \trianglelefteq y$ and $x \trianglelefteq w$, and therefore by the complete semifactorability of f

$$f(x, \operatorname{lcm}(y,w)) = f(x, yw/\gcd(y,w)) = f(x, yw/x) = f(x,y)f(x,w).$$

Thus by Definition 7.1 f is semifactorable. □

The following theorem presents a characterization of a completely semifactorable function, and it is essentially a reformulation of the definition of a completely semifactorable function.

Theorem 7.35. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely semifactorable if and only if*

(i) $\forall\, x \in \mathbb{Z}_+ : f(x,x) = 1,$

(ii) $\forall\, x,y,z,w \in \mathbb{Z}_+ : \Big[x \trianglelefteq z \trianglelefteq y \text{ and } zw = xy\Big]$
$\Rightarrow f(x,y) = f(x,z)f(x,w).$

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. Let us first assume that f is completely semifactorable. (i) By the complete semifactorability of f

$$\forall\, x \in \mathbb{Z}_+ : f(x,x) = 1.$$

(ii) Let $x,y,z,w \in \mathbb{Z}_+$ be such that $x \trianglelefteq z \trianglelefteq y$ and $zw = xy$. Then by Lemma 5.3 $x \trianglelefteq w$, and therefore by the complete semifactorability of f

$$f(x,y) = f(x, zw/x) = f(x,z)f(x,w).$$

Thus (i) and (ii) hold.

Let us next assume that (i) and (ii) hold. Let $x,y,w \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$ and $x \trianglelefteq w$. Then $w = xv$, where $v \in \mathbb{Z}_+$. Thus $yw = x \cdot yv$, and therefore $yw/x = yv$. Thus $y \trianglelefteq yw/x$, and therefore $x \trianglelefteq y \trianglelefteq yw/x$. Since $yw = x \cdot yw/x$, it follows by (ii) that $f(x, yw/x) = f(x,y)f(x,w)$. Thus f is completely semifactorable. □

Theorem 7.36. *The delta function $\delta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely semifactorable.*

Proof. Let $x,y,z,w \in \mathbb{Z}_+$ be such that $x \trianglelefteq z \trianglelefteq y$ and $zw = xy$. Let us assume that $x = y$. Since $x \trianglelefteq z \trianglelefteq y$ and by Lemma 5.3 $x \trianglelefteq w \trianglelefteq y$, it follows that $x = z$ and $x = w$, and therefore

$$\delta(x,y) = 1 = \delta(x,z)\delta(x,w).$$

Let us assume that $x \neq y$. Since $zw = xy$, it follows that $x = z$ if and only if $y = w$. Thus $\delta(x,z) = 1$ if and only if $\delta(x,w) = 0$, and therefore

$$\delta(x,y) = 0 = \delta(x,z)\delta(x,w).$$

Thus by Theorem 7.35 $\delta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely semifactorable. □

The following two theorems present, using primes, necessary and sufficient conditions for a semifactorable function to be completely semifactorable.

Theorem 7.37. *Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be semifactorable. Then f is completely semifactorable if and only if*

$$\forall\, x \in \mathbb{Z}_+ : \forall\, p \in \mathbb{P} : \forall\, n \in \mathbb{Z}_+ : f(x, xp^n) = f(x, xp)^n.$$

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be semifactorable. Let us first assume that f is completely semifactorable. Let $x \in \mathbb{Z}_+$ and $p \in \mathbb{P}$. Let us use induction on $n \in \mathbb{Z}_+$ to show that

$$\forall\, n \in \mathbb{Z}_+ : f(x, xp^n) = f(x, xp)^n.$$

If $n = 1$, then

$$f(x, xp^n) = f(x, xp) = f(x, xp)^n.$$

Let $n \in \mathbb{Z}_+$, and let us assume that $f(x, xp^n) = f(x, xp)^n$. Since $x \trianglelefteq xp^n$ and $x \trianglelefteq xp$, it follows by the complete semifactorability of f and the induction hypothesis

$$\begin{aligned} f(x, xp^{n+1}) &= f(x, (xp^n \cdot xp)/x) \\ &= f(x, xp^n) f(x, xp) \\ &= f(x, xp)^n f(x, xp) \\ &= f(x, xp)^{n+1}. \end{aligned}$$

Thus

$$\forall\, n \in \mathbb{Z}_+ : f(x, xp^n) = f(x, xp)^n.$$

Let us next assume that

$$\forall\, x \in \mathbb{Z}_+ : \forall\, p \in \mathbb{P} : \forall\, n \in \mathbb{Z}_+ : f(x, xp^n) = f(x, xp)^n.$$

(i) By the semifactorability of f

$$\forall\, x \in \mathbb{Z}_+ : f(x, x) = 1.$$

(ii) Let $x, y, w \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$ and $x \trianglelefteq w$. Then $y = xu$ and $w = xv$, where $u, v \in \mathbb{Z}_+$. Thus $yw/x = xuv$, and therefore by Theorem 7.5 and the assumption

$$\begin{aligned} f(x, yw/x) &= f(x, xuv) \\ &= \prod_{p \in \mathbb{P}} f(x, xp^{(xuv)(p) - x(p)}) \\ &= \prod_{p \in \mathbb{P}} f(x, xp)^{(xuv)(p) - x(p)} \\ &= \prod_{p \in \mathbb{P}} f(x, xp)^{x(p) + u(p) + v(p) - x(p)} \\ &= \prod_{p \in \mathbb{P}} f(x, xp)^{x(p) + u(p) - x(p) + x(p) + v(p) - x(p)} \\ &= \prod_{p \in \mathbb{P}} f(x, xp)^{[(xu)(p) - x(p)] + [(xv)(p) - x(p)]} \\ &= \prod_{p \in \mathbb{P}} f(x, xp)^{[y(p) - x(p)] + [w(p) - x(p)]} \end{aligned}$$

$$
\begin{aligned}
&= \prod_{p\in\mathbb{P}} f(x,xp)^{y(p)-x(p)} f(x,xp)^{w(p)-x(p)} \\
&= \prod_{p\in\mathbb{P}} f(x,xp^{y(p)-x(p)}) f(x,xp^{w(p)-x(p)}) \\
&= \Big[\prod_{p\in\mathbb{P}} f(x,xp^{y(p)-x(p)})\Big]\Big[\prod_{p\in\mathbb{P}} f(x,xp^{w(p)-x(p)})\Big] \\
&= f(x,y)f(x,w).
\end{aligned}
$$

By (i) and (ii) f is completely semifactorable. □

Theorem 7.38. *Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be semifactorable. Then f is completely semifactorable if and only if*

$$\forall\, x \in \mathbb{Z}_+ : \forall\, p \in \mathbb{P} : \forall\, n \in \mathbb{Z}_+ : f(x,xp^n) = f(x,xp)f(x,xp^{n-1}).$$

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be semifactorable. Let us first assume that f is completely semifactorable. Let $x \in \mathbb{Z}_+$, $p \in \mathbb{P}$, and $n \in \mathbb{Z}_+$. If $n = 1$, then by the complete semifactorability of f

$$f(x,xp^n) = f(x,xp) = f(x,xp)f(x,x) = f(x,xp)f(x,xp^{n-1}).$$

If $n > 1$, then by Theorem 7.37

$$f(x,xp^n) = f(x,xp)^n = f(x,xp)f(x,xp)^{n-1} = f(x,xp)f(x,xp^{n-1}).$$

Let us next assume that

$$\forall\, x \in \mathbb{Z}_+ : \forall\, p \in \mathbb{P} : \forall\, n \in \mathbb{Z}_+ : f(x,xp^n) = f(x,xp)f(x,xp^{n-1}).$$

Let $x \in \mathbb{Z}_+$ and $p \in \mathbb{P}$. Let us use induction on $n \in \mathbb{Z}_+$ to show that

$$\forall\, n \in \mathbb{Z}_+ : f(x,xp^n) = f(x,xp)^n.$$

If $n = 1$, then

$$f(x,xp^n) = f(x,xp) = f(x,xp)^n.$$

Let $n \in \mathbb{Z}_+$, and let us assume that $f(x,xp^n) = f(x,xp)^n$. Then by the assumption and the induction hypothesis

$$f(x,xp^{n+1}) = f(x,xp)f(x,xp^n) = f(x,xp)f(x,xp)^n = f(x,xp)^{n+1}.$$

Thus

$$\forall\, n \in \mathbb{Z}_+ : f(x,xp^n) = f(x,xp)^n,$$

and therefore

$$\forall\, x \in \mathbb{Z}_+ : \forall\, p \in \mathbb{P} : \forall\, n \in \mathbb{Z}_+ : f(x,xp^n) = f(x,xp)^n.$$

Thus by Theorem 7.37 f is completely semifactorable. □

The following theorem presents a prime related characterization of a completely semifactorable function, and therefore, in other words, it gives a necessary and sufficient condition for a function to be completely semifactorable.

Theorem 7.39. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely semifactorable if and only if*

(i) $\forall\, x \in \mathbb{Z}_+ : f(x,x) = 1$,

(ii) $\forall\, x, y \in \mathbb{Z}_+ : x \trianglelefteq y \Rightarrow f(x,y) = \prod_{p\in\mathbb{P}} f(x,xp)^{y(p)-x(p)}$,
where $x = \prod_{p\in\mathbb{P}} p^{x(p)}$ *and* $y = \prod_{p\in\mathbb{P}} p^{y(p)}$.

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. Let us first assume that f is completely semifactorable. (i) By the complete semifactorability of f

$$\forall\, x \in \mathbb{Z}_+ : f(x,x) = 1.$$

(ii) Let $x, y \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$, where $x = \prod_{p\in\mathbb{P}} p^{x(p)}$ and $y = \prod_{p\in\mathbb{P}} p^{y(p)}$. Let us note that if $y(p) - x(p) = 0$, then by the complete semifactorability of f

$$f(x, xp^{y(p)-x(p)}) = f(x,x) = 1 = f(x,xp)^{y(p)-x(p)}.$$

Thus by Theorems 7.34, 7.5, and 7.37

$$f(x,y) = \prod_{p\in\mathbb{P}} f(x, xp^{y(p)-x(p)}) = \prod_{p\in\mathbb{P}} f(x,xp)^{y(p)-x(p)}.$$

Thus (i) and (ii) hold.

Let us next assume that (i) and (ii) hold. Let $x, y, z, w \in \mathbb{Z}_+$ be such that $x \trianglelefteq z \trianglelefteq y$ and $zw = xy$, where $x = \prod_{p\in\mathbb{P}} p^{x(p)}$, $y = \prod_{p\in\mathbb{P}} p^{y(p)}$, $z = \prod_{p\in\mathbb{P}} p^{z(p)}$, and $w = \prod_{p\in\mathbb{P}} p^{w(p)}$. Since $zw = xy$, it follows that $z(p) + w(p) = x(p) + y(p)$ for all $p \in \mathbb{P}$, and therefore by (ii)

$$\begin{aligned}
f(x,y) &= \prod_{p\in\mathbb{P}} f(x,xp)^{y(p)-x(p)} \\
&= \prod_{p\in\mathbb{P}} f(x,xp)^{x(p)-x(p)+y(p)-x(p)} \\
&= \prod_{p\in\mathbb{P}} f(x,xp)^{z(p)-x(p)+w(p)-x(p)} \\
&= \prod_{p\in\mathbb{P}} f(x,xp)^{z(p)-x(p)} f(x,xp)^{w(p)-x(p)} \\
&= \Big[\prod_{p\in\mathbb{P}} f(x,xp)^{z(p)-x(p)}\Big]\Big[\prod_{p\in\mathbb{P}} f(x,xp)^{w(p)-x(p)}\Big] \\
&= f(x,z)f(x,w).
\end{aligned}$$

Thus by Theorem 7.35 f is completely semifactorable. $\square$

The following theorem presents a characterization of a completely semifactorable function that is closely related to Theorem 7.35.

Theorem 7.40. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely semifactorable if and only if*

(i) $\forall\, x \in \mathbb{Z}_+ : f(x,x) = 1,$

(ii) $\forall\, g, h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq] : f(g * h) = (fg) * (fh).$

Proof. Follows by Theorems 7.35 and 5.27. □

Definition 7.8. Let $g, h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. The D-convolution of g and h, that is $g * h$, is *D-discriminative* if

$$\forall\, x, y \in \mathbb{Z}_+ : \\ (g * h)(x, xy) = g(x,x)h(x,xy) + g(x,xy)h(x,x) \Rightarrow (y = 1 \text{ or } y \in \mathbb{P}).$$

Remark. A property of primes is that

$$\forall\, g, h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq] : \forall\, x, y \in \mathbb{Z}_+ : \\ y \in \mathbb{P} \Rightarrow (g * h)(x, xy) = g(x,x)h(x,xy) + g(x,xy)h(x,x).$$

Remark. The notion of D-discriminative D-convolution of arithmetic incidence functions is generalized from the notion of semidiscriminative Dirichlet convolution of arithmetic functions (see Definition 3.27).

Remark. If $g * h$ is D-discriminative, then by Theorem 5.17 also $h * g$ is D-discriminative.

Theorem 7.41. *The D-convolution $\zeta * \zeta$, where $\zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, is D-discriminative.*

Proof. Let $x, y \in \mathbb{Z}_+$, where $y \neq 1$ and $y \notin \mathbb{P}$. Let $u, v \in \mathbb{Z}_+$ be such that $1 < u < y$, $1 < v < y$, and $y = uv$. Then $x \lhd xu \lhd xy$ and $xu \cdot xv = x \cdot xy$. Thus by Lemma 5.3

$$\begin{aligned}(\zeta * \zeta)(x, xy) &= \sum_{\substack{x \trianglelefteq z \trianglelefteq xy \\ zw = x \cdot xy}} \zeta(x,z)\zeta(x,w) \\ &= \zeta(x,x)\zeta(x,xy) + \zeta(x,xy)\zeta(x,x) + \sum_{\substack{x \lhd z \lhd xy \\ zw = x \cdot xy}} \zeta(x,z)\zeta(x,w),\end{aligned}$$

where

$$\sum_{\substack{x \lhd z \lhd xy \\ zw = x \cdot xy}} \zeta(x,z)\zeta(x,w) > 0,$$

and therefore $(\zeta * \zeta)(x, xy) \neq \zeta(x,x)\zeta(x,xy) + \zeta(x,xy)\zeta(x,x)$. Thus by the contraposition principle

$$\forall\, x, y \in \mathbb{Z}_+ : \\ (\zeta * \zeta)(x, xy) = \zeta(x,x)\zeta(x,xy) + \zeta(x,xy)\zeta(x,x) \Rightarrow (y = 1 \text{ or } y \in \mathbb{P}),$$

and therefore by Definition 7.8 $\zeta * \zeta$ is D-discriminative. □

The following theorems present characterizations of a completely semifactorable function.

Theorem 7.42. *A function* $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ *is completely semifactorable if and only if*

(i) $\forall\, x \in \mathbb{Z}_+ : f(x,x) = 1,$

(ii) $\exists\, g, h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq] : f(g*h) = (fg)*(fh)$*, where* $g*h$ *is D-discriminative.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. Let us first assume that f is completely semifactorable. (i) By the complete semifactorability of f

$$\forall\, x \in \mathbb{Z}_+ : f(x,x) = 1.$$

(ii) By Theorem 7.40 $f(\zeta * \zeta) = (f\zeta) * (f\zeta)$, where (by Theorem 7.41) $\zeta * \zeta$ is D-discriminative. Thus (i) and (ii) hold.

Let us next assume that (i) and (ii) hold. Let $x, y \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$, where $x = \prod_{p\in\mathbb{P}} p^{x(p)}$ and $y = \prod_{p\in\mathbb{P}} p^{y(p)}$. Let us use induction on $\Omega(x,y) \in \mathbb{N}$, where $\Omega(x,y)$ stands for the total number of prime factors of y/x (see Definition 5.7), to show that

$$f(x,y) = \prod_{p\in\mathbb{P}} f(x,xp)^{y(p)-x(p)}.$$

If $\Omega(x,y) = 0$, then $x(p) = y(p)$ for all $p \in \mathbb{P}$, and therefore by (i)

$$f(x,y) = f(x,x) = 1 = \prod_{p\in\mathbb{P}} f(x,xp)^{y(p)-x(p)}.$$

If $\Omega(x,y) = 1$, then $y = xp$, where $p \in \mathbb{P}$ and $y(p) = x(p) + 1$, and therefore by (i)

$$f(x,y) = f(x,xp) = \prod_{p\in\mathbb{P}} f(x,xp)^{y(p)-x(p)}.$$

Let $\Omega(x,y) = n > 1$, and let us assume that

$$\forall\, u \in \mathbb{Z}_+ : (x \trianglelefteq u \trianglelefteq y \text{ and } \Omega(x,u) < n) \Rightarrow f(x,u) = \prod_{p\in\mathbb{P}} f(x,xp)^{u(p)-x(p)},$$

where $u = \prod_{p\in\mathbb{P}} p^{u(p)}$. Let us assume that $g, h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ satisfy the condition (ii), i.e. $g * h$ is D-discriminative and $f(g * h) = (fg) * (fh)$. Then by

Lemma 5.3, (i), and the induction hypothesis

$$\begin{aligned}
[(fg)*(fh)](x,y) &= \sum_{\substack{x\trianglelefteq z\trianglelefteq y\\ zw=xy}} (fg)(x,z)(fh)(x,w)\\
&= \sum_{\substack{x\trianglelefteq z\trianglelefteq y\\ zw=xy}} f(x,z)g(x,z)f(x,w)h(x,w)\\
&\overset{(1)}{=} f(x,x)g(x,x)f(x,y)h(x,y)\\
&\quad + f(x,y)g(x,y)f(x,x)h(x,x)\\
&\quad + \sum_{\substack{x\triangleleft z\triangleleft y\\ zw=xy}} f(x,z)g(x,z)f(x,w)h(x,w)\\
&\overset{(2)}{=} f(x,y)[g(x,x)h(x,y)+g(x,y)h(x,x)]\\
&\quad + \sum_{\substack{x\triangleleft z\triangleleft y\\ zw=xy}} \Big[\Big[\prod_{p\in\mathbb{P}} f(x,xp)^{z(p)-x(p)}\Big]g(x,z)\\
&\qquad\qquad \times \Big[\prod_{p\in\mathbb{P}} f(x,xp)^{w(p)-x(p)}\Big]h(x,w)\Big]\\
&= f(x,y)[g(x,x)h(x,y)+g(x,y)h(x,x)]\\
&\quad + \sum_{\substack{x\triangleleft z\triangleleft y\\ zw=xy}} \Big[\prod_{p\in\mathbb{P}} f(x,xp)^{z(p)-x(p)+w(p)-x(p)}\Big]g(x,z)h(x,w)\\
&= f(x,y)[g(x,x)h(x,y)+g(x,y)h(x,x)]\\
&\quad + \sum_{\substack{x\triangleleft z\triangleleft y\\ zw=xy}} \Big[\prod_{p\in\mathbb{P}} f(x,xp)^{x(p)-x(p)+y(p)-x(p)}\Big]g(x,z)h(x,w)\\
&= f(x,y)[g(x,x)h(x,y)+g(x,y)h(x,x)]\\
&\quad + \sum_{\substack{x\triangleleft z\triangleleft y\\ zw=xy}} \Big[\prod_{p\in\mathbb{P}} f(x,xp)^{y(p)-x(p)}\Big]g(x,z)h(x,w)\\
&= f(x,y)[g(x,x)h(x,y)+g(x,y)h(x,x)]\\
&\quad + \prod_{p\in\mathbb{P}} f(x,xp)^{y(p)-x(p)} \sum_{\substack{x\triangleleft z\triangleleft y\\ zw=xy}} g(x,z)h(x,w),
\end{aligned}$$

where (1) means "by Lemma 5.3",
(2) means "by (i) and the induction hypothesis".

On the other hand, by Lemma 5.3

$$\begin{aligned}
[f(g*h)](x,y) &= f(x,y)[g(x,x)h(x,y)+g(x,y)h(x,x)]\\
&\quad + f(x,y)\sum_{\substack{x\triangleleft z\triangleleft y\\ zw=xy}} g(x,z)h(x,w),
\end{aligned}$$

and therefore, since $f(g*h) = (fg)*(fh)$, it follows that

$$f(x,y)\sum_{\substack{x\lhd z\lhd y\\ zw=xy}} g(x,z)h(x,w) = \prod_{p\in\mathbb{P}} f(x,xp)^{y(p)-x(p)} \sum_{\substack{x\lhd z\lhd y\\ zw=xy}} g(x,z)h(x,w).$$

Since $\Omega(x,y) = n > 1$, it follows that $y = xu$, where $u \in \mathbb{Z}_+$ is such that $u \neq 1$ and $u \notin \mathbb{P}$, and therefore, since $g*h$ is D-discriminative, it follows that

$$(g*h)(x,y) \neq g(x,x)h(x,y) + g(x,y)h(x,x).$$

On the other hand (by Lemma 5.3)

$$(g*h)(x,y) = g(x,x)h(x,y) + g(x,y)h(x,x) + \sum_{\substack{x\lhd z\lhd y\\ zw=xy}} g(x,z)h(x,w),$$

and therefore

$$\sum_{\substack{x\lhd z\lhd y\\ zw=xy}} g(x,z)h(x,w) \neq 0.$$

Thus

$$f(x,y) = \prod_{p\in\mathbb{P}} f(x,xp)^{y(p)-x(p)},$$

and therefore

$$\forall\, x,y \in \mathbb{Z}_+ : x \unlhd y \Rightarrow f(x,y) = \prod_{p\in\mathbb{P}} f(x,xp)^{y(p)-x(p)},$$

where $x = \prod_{p\in\mathbb{P}} p^{x(p)}$ and $y = \prod_{p\in\mathbb{P}} p^{y(p)}$. Thus by Theorem 7.39 f is completely semifactorable. $\square$

Theorem 7.43. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \unlhd]$ is completely semifactorable if and only if*

(i) $\forall\, x \in \mathbb{Z}_+ : f(x,x) = 1$,

(ii) $f(\zeta * \zeta) = f * f$.

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \unlhd]$. Let us first assume that f is completely semifactorable. Let us note that by Theorem 5.9 (ii) is equivalent to $f(\zeta*\zeta) = (f\zeta)*(f\zeta)$. Thus by Theorem 7.40 (i) and (ii) hold.

Let us next assume that (i) and (ii) hold. Since by Theorem 7.41 $\zeta*\zeta$ is D-discriminative and by Theorem 5.9 $f(\zeta*\zeta) = (f\zeta)*(f\zeta)$, it follows by Theorem 7.42 that f is completely semifactorable. $\square$

Remark. Theorem 7.43 is a generalization of Theorem 3.40 to completely semifactorable arithmetic incidence functions. An example presented by K. L. Yocom [60, p. 120] demonstrates that Theorem 3.40, when applied to incidence functions, does not work as a characterization of a 'completely factorable' incidence function if the translation invariance of a function is not required (see Definition 4.16 and the related discussion).

Theorem 7.44. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely semifactorable if and only if*

(i) $\forall\, x \in \mathbb{Z}_+ : f(x,x) = 1,$

(ii) $\forall\, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq] : f(g * g) = (fg) * (fg).$

Proof. Follows by Theorems 7.40 and 7.43. □

Definition 7.9. Let $g, h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. The D-convolution of g and h, that is $g * h$, is *partially D-discriminative* if

$$\begin{aligned}&\forall\, x \in \mathbb{Z}_+ : \forall\, p \in \mathbb{P} : \forall\, n \in \mathbb{Z}_+ : \\ &\qquad (g * h)(x, xp^n) = g(x,x)h(x,xp^n) + g(x,xp^n)h(x,x) \Rightarrow n = 1.\end{aligned}$$

Remark. From Definition 7.8 it follows that if $g * h$ is D-discriminative, then it is also partially D-discriminative.

Remark. If $g * h$ is partially D-discriminative, then by Theorem 5.17 also $h * g$ is partially D-discriminative.

Theorem 7.45. *The D-convolution $\zeta * \mu$, where $\zeta, \mu \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, is not D-discriminative, but it is partially D-discriminative.*

Proof. Let $x, y \in \mathbb{Z}_+$ be such that

$$y = \prod_{\substack{i=1 \\ p_i \in \mathbb{P}}}^{k} p_i,$$

where $k \in \mathbb{Z}_+$, $k > 1$, k is odd, and $p_i \in \mathbb{P}$, $1 \leq i \leq k$, are distinct. Then by Theorems 5.29, 7.9, 7.5, and 5.30

$$\begin{aligned}\zeta(x,x)\mu(x,xy) + \zeta(x,xy)\mu(x,x) &= \mu(x,xy) + 1 \\ &= \Big[\prod_{p \in \mathbb{P}} \mu(x, xp^{(xy)(p) - x(p)})\Big] + 1 \\ &= \Big[\prod_{\substack{i=1 \\ p_i \in \mathbb{P}}}^{k} \mu(x, xp_i)\Big] + 1 \\ &= (-1)^k + 1 \\ &= 0 \\ &= (\zeta * \mu)(x, xy).\end{aligned}$$

Since $y \neq 1$ and $y \notin \mathbb{P}$, it follows by Definition 7.8 that $\zeta * \mu$ is not D-discriminative.

Let $x \in \mathbb{Z}_+$, $p \in \mathbb{P}$, and $n \in \mathbb{Z}_+$ be such that

$$(\zeta * \mu)(x, xp^n) = \zeta(x,x)\mu(x,xp^n) + \zeta(x,xp^n)\mu(x,x).$$

Then $\mu(x, xp^n) + 1 = 0$, and therefore $\mu(x, xp^n) = -1$. Let us assume that $n \neq 1$, i.e. $n \geq 2$. Since $\mu(x, xp^n) = \mu(xp^0, xp^{0+(n-2)+2})$, it follows by Theorem 5.30 that $\mu(x, xp^n) = 0$. This contradicts the fact that $\mu(x, xp^n) = -1$, and therefore $n = 1$. Thus

$$\begin{aligned} &\forall\, x \in \mathbb{Z}_+ : \forall\, p \in \mathbb{P} : \forall\, n \in \mathbb{Z}_+ : \\ &\qquad (\zeta * \mu)(x, xp^n) = \zeta(x, x)\mu(x, xp^n) + \zeta(x, xp^n)\mu(x, x) \Rightarrow n = 1. \end{aligned}$$

and therefore by Definition 7.9 $\zeta * \mu$ is partially D-discriminative. □

The following theorems present necessary and sufficient conditions for a semifactorable function to be completely semifactorable.

Theorem 7.46. *Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be semifactorable. Then f is completely semifactorable if and only if*

$$\exists\, g, h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq] : f(g * h) = (fg) * (fh),$$

where $g * h$ is partially D-discriminative.

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be semifactorable. Let us first assume that f is completely semifactorable. Then by Theorem 7.40 $f(\zeta * \mu) = (f\zeta) * (f\mu)$, where (by Theorem 7.45) $\zeta * \mu$ is partially D-discriminative.

Let us next assume that

$$\exists\, g, h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq] : f(g * h) = (fg) * (fh),$$

where $g * h$ is partially D-discriminative. (i) By the semifactorability of f

$$\forall\, x \in \mathbb{Z}_+ : f(x, x) = 1.$$

(ii) Let $x \in \mathbb{Z}_+$ and $p \in \mathbb{P}$. Let us use induction on $n \in \mathbb{Z}_+$ to show that

$$\forall\, n \in \mathbb{Z}_+ : f(x, xp^n) = f(x, xp)^n.$$

If $n = 1$, then

$$f(x, xp^n) = f(x, xp) = f(x, xp)^n.$$

Let $n > 1$, and let us assume that

$$\forall\, m \in \mathbb{Z}_+ : (m < n) \Rightarrow f(x, xp^m) = f(x, xp)^m.$$

Let us assume that $g, h \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ satisfy the assumption, i.e. $g * h$ is partially D-discriminative and $f(g * h) = (fg) * (fh)$. Let us note that if $z, w \in \mathbb{Z}_+$ are such that $x \triangleleft z \triangleleft xp^n$ and $zw = x \cdot xp^n$, then by Lemmas 5.13 and 5.3 $z = xp^{k_1}$ and $w = xp^{k_2}$, where $k_1, k_2 \in \mathbb{N}$ are such that $0 < k_1 < n$ and $0 < k_2 < n$. Since $z(p) = x(p) + k_1$ and $w(p) = x(p) + k_2$, it follows that $z = xp^{z(p)-x(p)}$ and $w = xp^{w(p)-x(p)}$, where $0 < z(p) - x(p) < n$ and

$0 < w(p) - x(p) < n$. Let us also note that if $z, w \in \mathbb{Z}_+$ are such that $zw = x \cdot xp^n$, then $z(p) + w(p) = x(p) + (xp^n)(p) = x(p) + x(p) + n$. Then by Lemma 5.3, the semifactorability of f, and the induction hypothesis

$$\begin{aligned}
[(fg) * (fh)](x, xp^n) &= \sum_{\substack{x \trianglelefteq z \trianglelefteq xp^n \\ zw = x \cdot xp^n}} (fg)(x, z)(fh)(x, w) \\
&= \sum_{\substack{x \trianglelefteq z \trianglelefteq xp^n \\ zw = x \cdot xp^n}} f(x, z)g(x, z)f(x, w)h(x, w) \\
&\overset{(1)}{=} f(x, x)g(x, x)f(x, xp^n)h(x, xp^n) \\
&\quad + f(x, xp^n)g(x, xp^n)f(x, x)h(x, x) \\
&\quad + \sum_{\substack{x \triangleleft z \triangleleft xp^n \\ zw = x \cdot p^n}} f(x, z)g(x, z)f(x, w)h(x, w) \\
&\overset{(2)}{=} f(x, xp^n)[g(x, x)h(x, xp^n) + g(x, xp^n)h(x, x)] \\
&\quad + \sum_{\substack{x \triangleleft z \triangleleft xp^n \\ zw = x \cdot xp^n}} f(x, xp^{z(p)-x(p)})g(x, z)f(x, xp^{w(p)-x(p)})h(x, w) \\
&\overset{(3)}{=} f(x, xp^n)[g(x, x)h(x, xp^n) + g(x, xp^n)h(x, x)] \\
&\quad + \sum_{\substack{x \triangleleft z \triangleleft xp^n \\ zw = x \cdot xp^n}} f(x, xp)^{z(p)-x(p)}g(x, z)f(x, xp)^{w(p)-x(p)}h(x, w) \\
&= f(x, xp^n)[g(x, x)h(x, xp^n) + g(x, xp^n)h(x, x)] \\
&\quad + \sum_{\substack{x \triangleleft z \triangleleft xp^n \\ zw = x \cdot xp^n}} f(x, xp)^{z(p)-x(p)+w(p)-x(p)}g(x, z)h(x, w) \\
&= f(x, xp^n)[g(x, x)h(x, xp^n) + g(x, xp^n)h(x, x)] \\
&\quad + \sum_{\substack{x \triangleleft z \triangleleft xp^n \\ zw = x \cdot xp^n}} f(x, xp)^{x(p)-x(p)+(xp^n)(p)-x(p)}g(x, z)h(x, w) \\
&= f(x, xp^n)[g(x, x)h(x, xp^n) + g(x, xp^n)h(x, x)] \\
&\quad + \sum_{\substack{x \triangleleft z \triangleleft xp^n \\ zw = x \cdot xp^n}} f(x, xp)^n g(x, z)h(x, w) \\
&= f(x, xp^n)[g(x, x)h(x, xp^n) + g(x, xp^n)h(x, x)] \\
&\quad + f(x, xp)^n \sum_{\substack{x \triangleleft z \triangleleft xp^n \\ zw = x \cdot xp^n}} g(x, z)h(x, w),
\end{aligned}$$

where (1) means "by Lemma 5.3",
(2) means "by the semifactorability of f",
(3) means "by the induction hypothesis".

On the other hand, by Lemma 5.3

$$\begin{aligned}[f(g*h)](x,xp^n) &= f(x,xp^n)[g(x,x)h(x,xp^n)+g(x,xp^n)h(x,x)] \\ &\quad + f(x,xp^n)\sum_{\substack{x\lhd z\lhd xp^n\\ zw=x\cdot xp^n}} g(x,z)h(x,w),\end{aligned}$$

and therefore, since $f(g*h)=(fg)*(fh)$, it follows that

$$f(x,xp^n)\sum_{\substack{x\lhd z\lhd xp^n\\ zw=x\cdot xp^n}} g(x,z)h(x,w) = f(x,xp)^n \sum_{\substack{x\lhd z\lhd xp^n\\ zw=x\cdot xp^n}} g(x,z)h(x,w).$$

Since $n>1$ and $g*h$ is partially D-discriminative , it follows that

$$(g*h)(x,xp^n)\neq g(x,x)h(x,xp^n)+g(x,xp^n)h(x,x).$$

On the other hand (by Lemma 5.3)

$$(g*h)(x,xp^n) = g(x,x)h(x,xp^n)+g(x,xp^n)h(x,x)+\sum_{\substack{x\lhd z\lhd xp^n\\ zw=x\cdot xp^n}} g(x,z)h(x,w),$$

and therefore

$$\sum_{\substack{x\lhd z\lhd xp^n\\ zw=x\cdot xp^n}} g(x,z)h(x,w)\neq 0.$$

Thus $f(x,xp^n)=f(x,xp)^n$, and therefore

$$\forall\, x\in\mathbb{Z}_+ : \forall\, p\in\mathbb{P} : \forall\, n\in\mathbb{Z}_+ : f(x,xp^n)=f(x,xp)^n.$$

Thus by Theorem 7.37 f is completely semifactorable. $\square$

Theorem 7.47. *Let $f\in\mathbb{I}[\mathbb{Z}_+,\unlhd]$ be semifactorable. Then f is completely semifactorable if and only if*

$$f^{*-1}=f\mu.$$

Proof. Let $f\in\mathbb{I}[\mathbb{Z}_+,\unlhd]$ be semifactorable. Let us first assume that f is completely semifactorable. Then by Theorem 7.40 $f(\zeta*\mu)=(f\zeta)*(f\mu)$, and therefore by Theorem 5.9 and Lemma 7.4 $f*(f\mu)=\delta$. Since $f*(f\mu)=\delta$, it follows by Theorem 5.17 that $(f\mu)*f=\delta$. Thus $f\mu$ is the D-convolution inverse of f, i.e. $f^{*-1}=f\mu$.

Let us next assume that $f^{*-1}=f\mu$. Since by Theorem 7.45 $\zeta*\mu$ is partially D-discriminative and from $f*(f\mu)=\delta$ it follows by Theorem 5.9 and Lemma 7.4 that $f(\zeta*\mu)=(f\zeta)*(f\mu)$, it follows by Theorem 7.46 that f is completely semifactorable. $\square$

Remark. The sufficiency of the condition $f^{*-1}=f\mu$ for the complete semifactorability of a semifactorable $f\in\mathbb{I}[\mathbb{Z}_+,\unlhd]$ in Theorem 7.47 can also be established by using Lemma 5.14 and Theorem 7.38.

Theorem 7.48. *Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be semifactorable. Then f is completely semifactorable if and only if*

$$\forall\, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq] : \Big[\forall\, x \in \mathbb{Z}_+ : g(x,x) \neq 0 \Rightarrow (fg)^{*-1} = fg^{*-1}\Big].$$

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be semifactorable. Let us first assume that f is completely semifactorable. Let $g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be such that

$$\forall\, x \in \mathbb{Z}_+ : g(x,x) \neq 0.$$

Then by Theorem 5.20 g has a D-convolution inverse, and therefore by Theorems 7.40 and 5.18

$$(fg) * (fg^{*-1}) = f(g * g^{*-1}) = f\delta.$$

From the complete semifactorability of f it follows by Lemma 7.4 that $f\delta = \delta$, and therefore $(fg) * (fg^{*-1}) = \delta$. Since $(fg) * (fg^{*-1}) = \delta$, it follows by Theorem 5.17 that $(fg^{*-1}) * (fg) = \delta$. Thus fg^{*-1} is the D-convolution inverse of fg, i.e. $(fg)^{*-1} = fg^{*-1}$.

Let us next assume that

$$\forall\, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq] : \Big[\forall\, x \in \mathbb{Z}_+ : g(x,x) \neq 0 \Rightarrow (fg)^{*-1} = fg^{*-1}\Big].$$

Since

$$\forall\, x \in \mathbb{Z}_+ : \zeta(x,x) \neq 0,$$

it follows by Theorems 5.20 and 5.9, and the assumption that

$$f^{*-1} = (f\zeta)^{*-1} = f\zeta^{*-1} = f\mu.$$

Thus by Theorem 7.47 f is completely semifactorable. $\square$

Theorem 7.49. *If $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely semifactorable, then*

$$\forall\, x \in \mathbb{Z}_+ : \forall\, p \in \mathbb{P} : \forall\, m, n \in \mathbb{N} : f^{*-1}(xp^n, xp^{n+m+2}) = 0.$$

Proof. Follows by Theorems 7.34, 7.47 and 5.30. $\square$

Theorem 7.50. *Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be semifactorable. Then f is completely semifactorable if and only if*

$$\forall\, x \in \mathbb{Z}_+ : \forall\, p \in \mathbb{P} : \forall\, n \in \mathbb{N} : f^{*-1}(x, xp^{n+2}) = 0.$$

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be semifactorable. Let us first assume that f is completely semifactorable. Let $x \in \mathbb{Z}_+$, $p \in \mathbb{P}$, and $n \in \mathbb{N}$. Then by Theorem 7.49 $f^{*-1}(x, xp^{n+2}) = 0$.

Let us next assume that

$$\forall\, x \in \mathbb{Z}_+ : \forall\, p \in \mathbb{P} : \forall\, n \in \mathbb{N} : f^{*-1}(x, xp^{n+2}) = 0.$$

Let $x \in \mathbb{Z}_+$ and $p \in \mathbb{P}$. Let us use induction on $n \in \mathbb{Z}_+$ to show that

$$\forall\, n \in \mathbb{Z}_+ : f(x, xp^n) = f(x, xp)^n.$$

If $n = 1$, then

$$f(x, xp^n) = f(x, xp) = f(x, xp)^n.$$

Let $n > 1$, and let us assume that

$$\forall\, m \in \mathbb{Z}_+ : (m < n) \Rightarrow f(x, xp^m) = f(x, xp)^m.$$

Then by Lemma 5.13, Theorem 7.8, the assumption, the induction hypothesis, and Lemma 5.7

$$\begin{aligned}
0 &= \delta(x, xp^n) \\
&= (f^{*-1} * f)(x, xp^n) \\
&= \sum_{\substack{x \trianglelefteq z \trianglelefteq xp^n \\ zw = x \cdot xp^n}} f^{*-1}(x, z) f(x, w) \\
&= f^{*-1}(x, x) f(x, xp^n) + f^{*-1}(x, xp) f(x, xp^{n-1}) \\
&= f(x, xp^n) - f(x, xp) f(x, xp)^{n-1} \\
&= f(x, xp^n) - f(x, xp)^n.
\end{aligned}$$

Thus $f(x, xp^n) = f(x, xp)^n$, and therefore

$$\forall\, x \in \mathbb{Z}_+ : \forall\, p \in \mathbb{P} : \forall\, n \in \mathbb{Z}_+ : f(x, xp^n) = f(x, xp)^n.$$

Thus by Theorem 7.37 f is completely semifactorable. $\square$

The D-convolution does not necessarily preserve the complete semifactorability. In order to verify this, let us investigate $\zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, which by Theorem 7.33 is completely semifactorable, and its D-convolution with itself, that is,

$$(\zeta * \zeta)(x, y) = \sum_{\substack{x \trianglelefteq z \trianglelefteq y \\ zw = xy}} \zeta(x, z)\zeta(x, w) = \sum_{x \trianglelefteq z \trianglelefteq y} 1 = \#\, [x, y]\,.$$

Let $p_1, p_2 \in \mathbb{P}$ be distinct. Then

$$\begin{aligned}
\left[1, p_1^2 p_2^2\right] &= \{1, p_1, p_2, p_1^2, p_1 p_2, p_2^2, p_1^2 p_2, p_1 p_2^2, p_1^2 p_2^2\}, \\
[1, p_1] &= \{1, p_1\}, \\
\left[1, p_1 p_2^2\right] &= \{1, p_1, p_2, p_1 p_2, p_2^2, p_1 p_2^2\}.
\end{aligned}$$

Thus $1 \trianglelefteq p_1$ and $1 \trianglelefteq p_1 p_2^2$ but

$$(\zeta * \zeta)(1, p_1^2 p_2^2) \neq (\zeta * \zeta)(1, p_1)(\zeta * \zeta)(1, p_1 p_2^2),$$

and therefore $\zeta * \zeta$ is not completely semifactorable.

Correspondingly, the C-convolution does not necessarily preserve the complete semifactorability. Let us now investigate $\zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ and its C-convolution with itself, that is,

$$\begin{aligned}
(\zeta *_C \zeta)(x,y) &= \sum_{\substack{\operatorname{gcf}(z,w)=x\\ \operatorname{lcm}(z,w)=y}} \zeta(x,z)\zeta(x,w)\\
&= \sum_{\substack{\operatorname{gcf}(z,w)=x\\ \operatorname{lcm}(z,w)=y}} 1\\
&= \#\{\langle z,w\rangle \mid \operatorname{gcf}(z,w)=x \text{ and } \operatorname{lcm}(z,w)=y\}.
\end{aligned}$$

Let $p_1, p_2 \in \mathbb{P}$ be distinct. Then

$$\begin{aligned}
(\zeta *_C \zeta)(1, p_1^2p_2^2) &= \#\left\{\left\langle 1, p_1^2p_2^2\right\rangle, \left\langle p_1^2, p_2^2\right\rangle, \left\langle p_2^2, p_1^2\right\rangle, \left\langle p_1^2p_2^2, 1\right\rangle\right\},\\
(\zeta *_C \zeta)(1, p_1) &= \#\left\{\langle 1, p_1\rangle, \langle p_1, 1\rangle\right\},\\
(\zeta *_C \zeta)(1, p_1p_2^2) &= \#\left\{\left\langle 1, p_1p_2^2\right\rangle, \left\langle p_1, p_2^2\right\rangle, \left\langle p_2^2, p_1\right\rangle, \left\langle p_1p_2^2, 1\right\rangle\right\}.
\end{aligned}$$

Thus $1 \trianglelefteq p_1$ and $1 \trianglelefteq p_1p_2^2$ but

$$(\zeta *_C \zeta)(1, p_1^2p_2^2) \neq (\zeta *_C \zeta)(1, p_1)(\zeta *_C \zeta)(1, p_1p_2^2),$$

and therefore $\zeta *_C \zeta$ is not completely semifactorable.

The D-convolution inverse of a completely semifactorable function is not necessarily completely semifactorable. In order to verify this, let us investigate the D-convolution inverse of $\zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, that is, $\mu \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. Let $p \in \mathbb{P}$. Then by Theorem 5.30 $\mu(1,p) = -1$ and $\mu(1,p^2) = 0$. Thus $1 \trianglelefteq p$ but

$$\mu(1,p^2) \neq \mu(1,p)\mu(1,p),$$

and therefore μ is not completely semifactorable.

Correspondingly, the C-convolution inverse of a completely semifactorable function is not necessarily completely semifactorable. Let us now investigate $\zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ and its C-convolution inverse, that is, $\mu_c \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. Let $p \in \mathbb{P}$. Then by Theorem 5.48 $\mu_c(1,p) = -1$ and $\mu_c(1,p^2) = -1$. Thus $1 \trianglelefteq p$ but

$$\mu_c(1,p^2) \neq \mu_c(1,p)\mu_c(1,p),$$

and therefore μ_c is not completely semifactorable.

It is evident that the complete semifactorability and the complete translation invariance of an arithmetic incidence function are properties that do not depend on each other. Of these two properties, the complete translation invariance presents more systematic and widespread conditions for a function and its values to fulfill compared to the complete semifactorability which, in turn, can be characterized, in a sense, as a more local property. In light of this observation, a natural approach is to assume the complete translation invariance of a function and, using this assumption, study its complete semifactorability.

Let us next investigate briefly completely translation invariant arithmetic incidence functions and their complete semifactorability. The following theorem presents a necessary and sufficient condition for a completely translation invariant function to be completely semifactorable.

Theorem 7.51. *Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be completely translation invariant. Then f is completely semifactorable if and only if*

(i) $\forall\, x \in \mathbb{Z}_+ : f(x,x) = 1,$

(ii) $\forall\, x, y, z \in \mathbb{Z}_+ : x \trianglelefteq z \trianglelefteq y \Rightarrow f(x,y) = f(x,z)f(z,y).$

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be completely translation invariant. Let us first assume that f is completely semifactorable. (i) By the semifactorability of f

$$\forall\, x \in \mathbb{Z}_+ : f(x,x) = 1.$$

(ii) Let $x, y, z \in \mathbb{Z}_+$ be such that $x \trianglelefteq z \trianglelefteq y$. Since $zx \cdot y/z = xy$, it follows by Theorems 7.35 and 6.20 that

$$f(x,y) = f(x,z)f(x, x \cdot y/z) = f(x,z)f(z,y).$$

Thus (i) and (ii) hold.

Let us next assume that (i) and (ii) hold. Let $x, y, z, w \in \mathbb{Z}_+$ be such that $x \trianglelefteq z \trianglelefteq y$ and $zw = xy$. Then by (ii) and Theorem 6.20

$$f(x,y) = f(x,z)f(z,y) = f(x,z)f(x,w).$$

Thus by Theorem 7.35 f is completely semifactorable. □

Remark. Both the complete translation invariance and the complete semifactorability have an important role concerning the subsequent study concerning the complete factorability of an arithmetic incidence function, and Theorem 7.51 plays also a minor part in this study.

7.4 Complete Semicompressibility

The evident symmetry that holds between the semifactorability and the semicompressibility of an arithmetic incidence function suggests also a property that is analogous to the complete semifactorability of an arithmetic incidence function.

Definition 7.10. A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is *completely semicompressible* if

(i) $\forall\, x \in \mathbb{Z}_+ : f(x, x) = 1,$

(ii) $\forall\, x, y, z \in \mathbb{Z}_+ : \Big[x \trianglelefteq y,\ z \trianglelefteq y, \text{ and } y \trianglelefteq xz\Big]$
$\Rightarrow f(xz/y, y) = f(x, y)f(z, y).$

Lemma 7.7. *If $x, y, z \in \mathbb{Z}_+$ are such that $x \trianglelefteq y$, $z \trianglelefteq y$, and $y \trianglelefteq xz$, then $xz/y \trianglelefteq x$ and $xz/y \trianglelefteq z$.*

Proof. Elementary. □

Remark. If $x, y, z \in \mathbb{Z}_+$ are such that $x \trianglelefteq y$, $z \trianglelefteq y$, and $y \trianglelefteq xz$, then by Lemma 7.7 $xz/y \trianglelefteq y$.

The following theorem establishes that if an arithmetic incidence function is completely semicompressible, then it is necessarily also semicompressible.

Theorem 7.52. *If $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely semicompressible, then it is semicompressible.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be completely semicompressible. (i) By the complete semicompressibility of f

$$\forall\, x \in \mathbb{Z}_+ : f(x, x) = 1.$$

(ii) Let $x, y, z \in \mathbb{Z}_+$ be such that $y = \operatorname{lcm}(x, z)$. Then $x \trianglelefteq y$ and $z \trianglelefteq y$. Since xz is a common multiple of x and z, it follows by Theorem 2.15 that $y \trianglelefteq xz$, and therefore by the complete semicompressibility of f

$$f(\operatorname{gcf}(x, z), y) = f(xz/\operatorname{lcm}(x, z), y) = f(xz/y, y) = f(x, y)f(z, y).$$

Thus by Definition 7.6 f is semicompressible. □

The following theorem presents a characterization of a completely semicompressible function, and it is essentially a reformulation of the definition of a completely semicompressible function.

Theorem 7.53. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely semicompressible if and only if*

(i) $\forall\, x \in \mathbb{Z}_+ : f(x, x) = 1,$

(ii) $\forall\, x, y, z, w \in \mathbb{Z}_+ : \Big[x \trianglelefteq z \trianglelefteq y \text{ and } zw = xy\Big]$
$\Rightarrow f(x, y) = f(z, y)f(w, y).$

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. Let us first assume that f is completely semicompressible. (i) By the complete semicompressibility of f

$$\forall\, x \in \mathbb{Z}_+ : f(x,x) = 1.$$

(ii) Let $x, y, z, w \in \mathbb{Z}_+$ be such that $x \trianglelefteq z \trianglelefteq y$ and $zw = xy$. Then by Lemma 5.3 $w \trianglelefteq y$, and from $zw = xy$ it follows that $y \trianglelefteq zw$, and therefore by the complete semicompressibility of f

$$f(x,y) = f(zw/y, y) = f(z,y)f(w,y).$$

Thus (i) and (ii) hold.

Let us next assume that (i) and (ii) hold. Let $x, y, z \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$, $z \trianglelefteq y$, and $y \trianglelefteq xz$. Then by Lemma 7.7 $xz/y \trianglelefteq x$, and therefore $xz/y \trianglelefteq x \trianglelefteq y$. Since $xz = xz/y \cdot y$, it follows by (ii) that $f(xz/y, y) = f(x,y)f(z,y)$. Thus f is completely semicompressible. $\square$

The following two theorems present, using primes, necessary and sufficient conditions for a semicompressible function to be completely semicompressible.

Theorem 7.54. *Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be semicompressible. Then f is completely semicompressible if and only if*

$$\forall\, x \in \mathbb{Z}_+ : \forall\, p \in \mathbb{P} : \forall\, n \in \mathbb{Z}_+ : n \le x(p) \Rightarrow f(x/p^n, x) = f(x/p, x)^n,$$

where $x = \prod_{p \in \mathbb{P}} p^{x(p)}$.

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be semicompressible. Let us first assume that f is completely semicompressible. Let $x \in \mathbb{Z}_+$, where $x = \prod_{p \in \mathbb{P}} p^{x(p)}$. Let $p \in \mathbb{P}$, and let us use induction on $n \in \mathbb{Z}_+$ to show that

$$\forall\, n \in \mathbb{Z}_+ : n \le x(p) \Rightarrow f(x/p^n, x) = f(x/p, x)^n.$$

If $n = 1$, then

$$f(x/p^n, x) = f(x/p, x) = f(x/p, x)^n.$$

Let $n \in \mathbb{Z}_+$ be such that $n < x(p)$, and let us assume that

$$\forall\, n \in \mathbb{Z}_+ : n < x(p) \Rightarrow f(x/p^n, x) = f(x/p, x)^n.$$

Since $x/p^n \trianglelefteq x$, $x/p \trianglelefteq x$, and $x \trianglelefteq x/p^n \cdot x/p$, it follows by the complete semicompressibility of f and the induction hypothesis

$$\begin{aligned} f(x/p^{n+1}, x) &= f((x/p^n \cdot x/p)/x, x) \\ &= f(x/p^n, x) f(x/p, x) \\ &= f(x/p, x)^n f(x/p, x) \\ &= f(x/p, x)^{n+1}. \end{aligned}$$

Thus

$$\forall\, n \in \mathbb{Z}_+ : n \le x(p) \Rightarrow f(x/p^n, x) = f(x/p, x)^n.$$

Let us next assume that

$$\forall\, x \in \mathbb{Z}_+ : \forall\, p \in \mathbb{P} : \forall\, n \in \mathbb{Z}_+ : n \leq x(p) \Rightarrow f(x/p^n, x) = f(x/p, x)^n,$$

where $x = \prod_{p \in \mathbb{P}} p^{x(p)}$. (i) By the semicompressibility of f

$$\forall\, x \in \mathbb{Z}_+ : f(x, x) = 1.$$

(ii) Let $x, y, z \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$, $z \trianglelefteq y$, and $y \trianglelefteq xz$. Then $xz = yt$, where $t \in \mathbb{Z}_+$. Let us note that if $k \in \mathbb{N}$ is such that $k = 0$, then by (i)

$$f(y/p^k, y) = f(y, y) = 1.$$

Thus by Theorem 7.26, (i), and the assumption

$$\begin{aligned}
f(xz/y, y) &= f(t, y) \\
&= \prod_{p \in \mathbb{P}} f(y/p^{y(p)-t(p)}, y) \\
&= \prod_{\substack{p \in \mathbb{P} \\ y(p)-t(p)>0}} f(y/p, y)^{y(p)-t(p)} \\
&= \prod_{\substack{p \in \mathbb{P} \\ y(p)-t(p)>0}} f(y/p, y)^{y(p)-t(p)-(xz)(p)+(yt)(p)} \\
&= \prod_{\substack{p \in \mathbb{P} \\ y(p)-t(p)>0}} f(y/p, y)^{y(p)-t(p)-x(p)-z(p)+y(p)+t(p)} \\
&= \prod_{\substack{p \in \mathbb{P} \\ y(p)-x(p)+y(p)-z(p)>0}} f(y/p, y)^{[y(p)-x(p)]+[y(p)-z(p)]} \\
&= \prod_{\substack{p \in \mathbb{P} \\ y(p)-x(p)+y(p)-z(p)>0}} f(y/p, y)^{y(p)-x(p)} f(y/p, y)^{y(p)-z(p)} \\
&= \Big[\prod_{\substack{p \in \mathbb{P} \\ y(p)-x(p)>0}} f(y/p, y)^{y(p)-x(p)} \Big] \Big[\prod_{\substack{p \in \mathbb{P} \\ y(p)-z(p)>0}} f(y/p, y)^{y(p)-z(p)} \Big] \\
&= \Big[\prod_{p \in \mathbb{P}} f(y/p^{y(p)-x(p)}, y) \Big] \Big[\prod_{p \in \mathbb{P}} f(y/p^{y(p)-z(p)}, y) \Big] \\
&= f(x, y) f(z, y).
\end{aligned}$$

By (i) and (ii) f is completely semicompressible. □

Theorem 7.55. *Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be semicompressible. Then f is completely semicompressible if and only if*

$$\begin{aligned}
\forall\, x \in \mathbb{Z}_+ : \forall\, p \in \mathbb{P} : \forall\, n \in \mathbb{Z}_+ : &\Big[n \leq x(p) \Big] \\
&\Rightarrow f(x/p^n, x) = f(x/p, x) f(x/p^{n-1}, x),
\end{aligned}$$

where $x = \prod_{p \in \mathbb{P}} p^{x(p)}$.

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be semicompressible. Let us first assume that f is completely semicompressible. Let $x \in \mathbb{Z}_+$, where $x = \prod_{p\in\mathbb{P}} p^{x(p)}$, and let $p \in \mathbb{P}$, and let $n \in \mathbb{Z}_+$ be such that $n \leq x(p)$. If $n = 1$, then by the complete semicompressibility of f

$$f(x/p^n, x) = f(x/p, x) = f(x/p, x)f(x, x) = f(x/p, x)f(x/p^{n-1}, x).$$

If $n > 1$, then by Theorem 7.54

$$f(x/p^n, x) = f(x/p, x)^n = f(x/p, x)f(x/p, x)^{n-1} = f(x/p, x)f(x/p^{n-1}, x).$$

Let us next assume that

$$\forall\, x \in \mathbb{Z}_+ : \forall\, p \in \mathbb{P} : \forall\, n \in \mathbb{Z}_+ : \Big[n \leq x(p)\Big] \\ \Rightarrow f(x/p^n, x) = f(x/p, x)f(x/p^{n-1}, x),$$

where $x = \prod_{p\in\mathbb{P}} p^{x(p)}$. Let $x \in \mathbb{Z}_+$ and $p \in \mathbb{P}$, and let us use induction on $n \in \mathbb{Z}_+$ to show that

$$\forall\, n \in \mathbb{Z}_+ : n \leq x(p) \Rightarrow f(x/p^n, x) = f(x/p, x)^n.$$

If $n = 1$, then

$$f(x/p^n, x) = f(x/p, x) = f(x/p, x)^n.$$

Let $n \in \mathbb{Z}_+$ be such that $n < x(p)$, and let us assume that

$$\forall\, n \in \mathbb{Z}_+ : n < x(p) \Rightarrow f(x/p^n, x) = f(x/p, x)^n.$$

Then by the assumption and the induction hypothesis

$$f(x/p^{n+1}, x) = f(x/p, x)f(x/p^n, x) = f(x/p, x)f(x/p, x)^n = f(x/p, x)^{n+1}.$$

Thus

$$\forall\, n \in \mathbb{Z}_+ : n \leq x(p) \Rightarrow f(x/p^n, x) = f(x/p, x)^n,$$

and therefore

$$\forall\, x \in \mathbb{Z}_+ : \forall\, p \in \mathbb{P} : \forall\, n \in \mathbb{Z}_+ : n \leq x(p) \Rightarrow f(x/p^n, x) = f(x/p, x)^n.$$

Thus by Theorem 7.54 f is completely semicompressible. $\square$

The following theorem presents a prime related characterization of a completely semicompressible function, and therefore, in other words, it gives a necessary and sufficient condition for a function to be completely semicompressible.

Theorem 7.56. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely semicompressible if and only if*

(i) $\forall\, x \in \mathbb{Z}_+ : f(x, x) = 1,$

(ii) $\forall\, x, y \in \mathbb{Z}_+ : x \trianglelefteq y \Rightarrow f(x, y) = \prod_{\substack{p\in\mathbb{P}\\ p\trianglelefteq y}} f(y/p, y)^{y(p)-x(p)},$
where $x = \prod_{p\in\mathbb{P}} p^{x(p)}$ and $y = \prod_{p\in\mathbb{P}} p^{y(p)}$.

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. Let us first assume that f is completely semicompressible. (i) By the complete semicompressibility of f

$$\forall\, x \in \mathbb{Z}_+ : f(x, x) = 1.$$

(ii) Let $x, y \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$, where $x = \prod_{p\in\mathbb{P}} p^{x(p)}$ and $y = \prod_{p\in\mathbb{P}} p^{y(p)}$. Let us note that if $y(p) - x(p) = 0$, then by (i)

$$f(y/p^{y(p)-x(p)}, y) = f(y, y) = 1.$$

Thus by Theorems 7.52, 7.26, and 7.54

$$\begin{aligned} f(x, y) &= \prod_{p\in\mathbb{P}} f(y/p^{y(p)-x(p)}, y) \\ &= \prod_{\substack{p\in\mathbb{P} \\ y(p)-x(p)>0}} f(y/p^{y(p)-x(p)}, y) \\ &= \prod_{\substack{p\in\mathbb{P} \\ y(p)-x(p)>0}} f(y/p, y)^{y(p)-x(p)} \\ &= \prod_{\substack{p\in\mathbb{P} \\ p\trianglelefteq y}} f(y/p, y)^{y(p)-x(p)}. \end{aligned}$$

Thus (i) and (ii) hold.

Let us next assume that (i) and (ii) hold. Let $x, y, z, w \in \mathbb{Z}_+$ be such that $x \trianglelefteq z \trianglelefteq y$ and $zw = xy$, where $x = \prod_{p\in\mathbb{P}} p^{x(p)}$, $y = \prod_{p\in\mathbb{P}} p^{y(p)}$, $z = \prod_{p\in\mathbb{P}} p^{z(p)}$, and $w = \prod_{p\in\mathbb{P}} p^{w(p)}$. Since $zw = xy$, it follows that $z(p) + w(p) = x(p) + y(p)$ for all $p \in \mathbb{P}$, and therefore by (ii)

$$\begin{aligned} f(x, y) &= \prod_{\substack{p\in\mathbb{P} \\ p\trianglelefteq y}} f(y/p, y)^{y(p)-x(p)} \\ &= \prod_{\substack{p\in\mathbb{P} \\ p\trianglelefteq y}} f(y/p, y)^{y(p)-x(p)+y(p)-y(p)} \\ &= \prod_{\substack{p\in\mathbb{P} \\ p\trianglelefteq y}} f(y/p, y)^{y(p)-z(p)+y(p)-w(p)} \\ &= \prod_{\substack{p\in\mathbb{P} \\ p\trianglelefteq y}} f(y/p, y)^{y(p)-z(p)} f(y/p, y)^{y(p)-w(p)} \\ &= \Big[\prod_{\substack{p\in\mathbb{P} \\ p\trianglelefteq y}} f(y/p, y)^{y(p)-z(p)}\Big]\Big[\prod_{\substack{p\in\mathbb{P} \\ p\trianglelefteq y}} f(y/p, y)^{y(p)-w(p)}\Big] \\ &= f(z, y) f(w, y). \end{aligned}$$

Thus by Theorem 7.53 f is completely semicompressible. □

Theorem 7.57. *Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be completely translation invariant. Then f is completely semicompressible if and only if it is completely semifactorable.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be completely translation invariant. Let us first assume that f is completely semicompressible. Let $x, y, z, w \in \mathbb{Z}_+$ be such that $x \trianglelefteq z \trianglelefteq y$ and $zw = xy$. Then by Theorems 7.53 and 6.20

$$f(x, y) = f(z, y)f(w, y) = f(w, y)f(z, y) = f(x, z)f(x, w).$$

Thus by Theorem 7.35 f is completely semifactorable.

Let us next assume that f is completely semifactorable. Let $x, y, z, w \in \mathbb{Z}_+$ be such that $x \trianglelefteq z \trianglelefteq y$ and $zw = xy$. Then by Theorems 7.35 and 6.20

$$f(x, y) = f(x, z)f(x, w) = f(w, y)f(z, y) = f(z, y)f(w, y).$$

Thus by Theorem 7.53 f is completely semicompressible. $\square$

Let us next demonstrate the usefulness of Theorem 7.57. Since by Theorems 6.16 and 6.19 the functions $\zeta, \delta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ are completely translation invariant and by Theorems 7.33 and 7.36 they are completely semifactorable, it follows by Theorem 7.57 that they are also completely semicompressible.

Let us next demonstrate that the complete semifactorability and the complete semicompressibility of a function are properties that do not depend on each other. Let $p_1, p_2 \in \mathbb{P}$ be distinct, and let us define the functions $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ as follows:

$$f : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C} : f(x, y) = \begin{cases} 1 & \text{if } x = y, \\ 1 & \text{if } x = 1 \text{ and } y = p_1, \\ 1 & \text{if } x = 1 \text{ and } y = p_2, \\ 1 & \text{if } x = 1 \text{ and } y = p_1^2, \\ 1 & \text{if } x = 1 \text{ and } y = p_1p_2, \\ 1 & \text{if } x = 1 \text{ and } y = p_2^2, \\ 1 & \text{if } x = 1 \text{ and } y = p_1^2p_2, \\ 1 & \text{if } x = 1 \text{ and } y = p_1p_2^2, \\ 1 & \text{if } x = 1 \text{ and } y = p_1^2p_2^2, \\ 0 & \text{otherwise.} \end{cases}$$

$$g : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C} : g(x, y) = \begin{cases} 1 & \text{if } x = y, \\ 1 & \text{if } x = 1 \text{ and } y = p_1^2p_2^2, \\ 1 & \text{if } x = p_1 \text{ and } y = p_1^2p_2^2, \\ 1 & \text{if } x = p_2 \text{ and } y = p_1^2p_2^2, \\ 1 & \text{if } x = p_1^2 \text{ and } y = p_1^2p_2^2, \\ 1 & \text{if } x = p_1p_2 \text{ and } y = p_1^2p_2^2, \\ 1 & \text{if } x = p_2^2 \text{ and } y = p_1^2p_2^2, \\ 1 & \text{if } x = p_1^2p_2 \text{ and } y = p_1^2p_2^2, \\ 1 & \text{if } x = p_1p_2^2 \text{ and } y = p_1^2p_2^2, \\ 0 & \text{otherwise.} \end{cases}$$

Then, by Theorems 7.35 and 7.53, f is completely semifactorable but not completely semicompressible, whereas g is completely semicompressible but not completely semifactorable. This establishes that if a function is completely semifactorable, then it is not necessarily completely semicompressible. Correspondingly, if a function is completely semicompressible, then it is not necessarily completely semifactorable.

It is worth noting that the function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ defined above is nontrivially completely semifactorable and not merely semifactorable. Correspondingly, the function $g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ defined above is nontrivially completely semicompressible and not merely semicompressible. In fact, the functions $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ which were used to establish that the semifactorability and the semicompressibility are two independent properties, would do the same also for the complete semifactorability and the complete semicompressibility. However, in this case the semifactorable function f is trivially completely semifactorable, whereas the semicompressible function g is trivially completely semicompressible.

The D-convolution does not necessarily preserve the complete semicompressibility. In order to verify this, let us investigate $\zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, which by Theorems 6.16, 7.33, and 7.57 is completely semicompressible, and its D-convolution with itself, that is, $\zeta * \zeta$, which by Theorems 6.16 and 6.23 is completely translation invariant. Since, as demonstrated earlier, $\zeta * \zeta$ is not completely semifactorable, it follows by Theorem 7.57 that it is not completely semicompressible.

Correspondingly, the C-convolution does not necessarily preserve the complete semicompressibility. Let us now investigate $\zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ and its C-convolution with itself, that is, $\zeta *_C \zeta$. Since, as demonstrated earlier, $\zeta *_C \zeta$ is not completely semifactorable, it follows by Theorem 7.57 that it is not completely semicompressible.

The D-convolution inverse of a completely semicompressible function is not necessarily completely semicompressible. In order to verify this, let us investigate the D-convolution inverse of $\zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, that is, $\mu \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, which by Theorem 6.25 is completely translation invariant. Since, as demonstrated earlier, μ is not completely semifactorable, it follows by Theorem 7.57 that it is not completely semicompressible.

Correspondingly, the C-convolution inverse of a completely semicompressible function is not necessarily completely semicompressible. Let us now investigate $\zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ and its C-convolution inverse, that is, $\mu_c \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, which by Theorem 6.29 is completely translation invariant. Since, as demonstrated earlier, μ_c is not completely semifactorable, it follows by Theorem 7.57 that it is not completely semicompressible.

Chapter 8

Factorable Functions

8.1 Factorability

The notion of factorability of an arithmetic incidence function is, in a sense, another generalization of the notion of multiplicativity of an arithmetic function of one variable. While building essentially upon the notion of semifactorability of an arithmetic incidence function, the factorability introduces a stronger condition for a function to fulfill explaining the chosen terminology. In effect, the notion of factorability of an arithmetic incidence function is a specific application of the notion of factorability of an incidence function (see Definition 4.14).

Definition 8.1. A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is *factorable* if

(i) $\forall\, x \in \mathbb{Z}_+ : f(x,x) = 1$,

(ii) $\forall\, x,y,z,w \in \mathbb{Z}_+ : \Big[x \trianglelefteq y,\ z \trianglelefteq w,\ \text{and}\ \operatorname{gcf}(x,z) = \operatorname{gcf}(y,w)\Big]$
$\Rightarrow f(\operatorname{lcm}(x,z), \operatorname{lcm}(y,w)) = f(x,y)f(z,w)$.

Theorem 8.1. *The zeta function $\zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is factorable.*

Proof. (i) Let $x \in \mathbb{Z}_+$. Since $x \trianglelefteq x$, it follows that $\zeta(x,x) = 1$. (ii) Let $x,y,z,w \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$, $z \trianglelefteq w$, and $\operatorname{gcf}(x,z) = \operatorname{gcf}(y,w)$. Then $\operatorname{lcm}(x,z) \trianglelefteq \operatorname{lcm}(y,w)$, and therefore

$$\zeta(\operatorname{lcm}(x,z), \operatorname{lcm}(y,w)) = 1 = \zeta(x,y)\zeta(z,w).$$

By (i) and (ii) $\zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is factorable. $\square$

The following two theorems present properties of a factorable function, and therefore, in other words, they give necessary conditions that a function must fulfill in order to be factorable.

Theorem 8.2. *If $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is factorable, then it is translation invariant.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be factorable, and let $x, y, z \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$ and $\mathrm{gcf}(x, z) = \mathrm{gcf}(y, z)$. Since $x \trianglelefteq y$, $z \trianglelefteq z$, and $\mathrm{gcf}(x, z) = \mathrm{gcf}(y, z)$, it follows by the factorability of f that

$$f(\mathrm{lcm}(x, z), \mathrm{lcm}(y, z)) = f(x, y)f(z, z) = f(x, y).$$

Thus by Definition 6.2 f is translation invariant. □

Theorem 8.3. *If $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is factorable, then it is semifactorable.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be factorable. (i) By the factorability of f

$$\forall\, x \in \mathbb{Z}_+ : f(x, x) = 1.$$

(ii) Let $x, y, w \in \mathbb{Z}_+$ be such that $x = \mathrm{gcf}(y, w)$. Then $x \trianglelefteq y$, $x \trianglelefteq w$, and $\mathrm{gcf}(x, x) = \mathrm{gcf}(y, w)$, and therefore by the factorability of f

$$f(x, \mathrm{lcm}(y, w)) = f(\mathrm{lcm}(x, x), \mathrm{lcm}(y, w)) = f(x, y)f(x, w).$$

Thus by Definition 7.1 f is semifactorable. □

The following theorem presents sufficient condition for a function to be factorable, and it captures the effect of combining the properties of translation invariance and semifactorability.

Theorem 8.4. *If $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is translation invariant and semifactorable, then it is factorable.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be translation invariant and semifactorable. (i) By the semifactorability of f

$$\forall\, x \in \mathbb{Z}_+ : f(x, x) = 1.$$

(ii) Let $x, y, z, w \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$, $z \trianglelefteq w$, and $\mathrm{gcf}(x, z) = \mathrm{gcf}(y, w)$. Then by Lemma 5.8 $\mathrm{gcf}(x, z) = \mathrm{gcf}(y, z)$ and $\mathrm{gcf}(x, z) = \mathrm{gcf}(x, w)$, and therefore by the translation invariance of f it follows that

$$f(x, y) = f(\mathrm{lcm}(x, z), \mathrm{lcm}(y, z)) \quad \text{and} \quad f(z, w) = f(\mathrm{lcm}(x, z), \mathrm{lcm}(x, w)).$$

Since $y \trianglelefteq \mathrm{lcm}(y, z)$, it follows that $x \trianglelefteq \mathrm{lcm}(y, z)$. Thus by Theorems 2.17 and 2.19

$$\begin{aligned}
\mathrm{gcf}(\mathrm{lcm}(y, z), \mathrm{lcm}(x, w)) &= \mathrm{gcf}(\mathrm{lcm}(x, w), \mathrm{lcm}(y, z)) \\
&= \mathrm{lcm}(x, \mathrm{gcf}(w, \mathrm{lcm}(y, z))) \\
&= \mathrm{lcm}(x, \mathrm{gcf}(\mathrm{lcm}(z, y), w))) \\
&= \mathrm{lcm}(x, \mathrm{lcm}(z, \mathrm{gcf}(y, w))) \\
&= \mathrm{lcm}(x, \mathrm{lcm}(z, \mathrm{gcf}(x, z))) \\
&= \mathrm{lcm}(x, \mathrm{lcm}(z, \mathrm{gcf}(z, x))) \\
&= \mathrm{lcm}(x, z),
\end{aligned}$$

and therefore by the semifactorability of f and Theorem 2.17

$$\begin{aligned} f(x,y)f(z,w) &= f(\operatorname{lcm}(x,z),\operatorname{lcm}(y,z))f(\operatorname{lcm}(x,z),\operatorname{lcm}(x,w)) \\ &= f(\operatorname{lcm}(x,z),\operatorname{lcm}(\operatorname{lcm}(y,z),\operatorname{lcm}(x,w))) \\ &= f(\operatorname{lcm}(x,z),\operatorname{lcm}(\operatorname{lcm}(x,y),\operatorname{lcm}(z,w))) \\ &= f(\operatorname{lcm}(x,z),\operatorname{lcm}(y,w)). \end{aligned}$$

By (i) and (ii) f is factorable. □

Remark. Theorem 8.4 is a specific application of a general result presented by D. A. Smith [49, pp. 357–358], accompanied with a remark that the result depends only on the local distributivity of the underlying local lattice. The context of that remark considered, the local distributivity cannot be taken as a necessary condition but as a sufficient condition which comes along by the setting chosen by D. A. Smith. Consequently, the above proof of Theorem 8.4 utilizes only the modularity of the factor lattice $(\mathbb{Z}_+, \trianglelefteq)$, demonstrating that the stronger property of distributivity is not actually needed.

The following theorem combines the results of the three immediately preceding theorems and presents a characterization of a factorable function, and therefore, in other words, it gives a necessary and sufficient condition for a function to be factorable.

Theorem 8.5. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is factorable if and only if it is translation invariant and semifactorable.*

Proof. Follows by Theorems 8.2, 8.3, and 8.4. □

Theorem 8.6. *The delta function $\delta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is factorable.*

Proof. Follows by Theorems 6.6, 7.3, and 8.5. □

Theorem 8.7. *The Möbius function $\mu \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is factorable.*

Proof. Follows by Theorems 6.10, 7.9, and 8.5. □

Theorem 8.8. *The complementary Möbius function $\mu_c \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is factorable.*

Proof. Follows by Theorems 6.14, 7.13, and 8.5. □

The following theorem presents a prime related characterization of a factorable function.

Theorem 8.9. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is factorable if and only if*

(i) $\forall\, x \in \mathbb{Z}_+ : f(x,x) = 1,$

(ii) $\forall\, x, y \in \mathbb{Z}_+ : x \trianglelefteq y \Rightarrow f(x,y) = \prod_{p\in\mathbb{P}} f(p^{x(p)}, p^{y(p)}),$
where $x = \prod_{p\in\mathbb{P}} p^{x(p)}$ *and* $y = \prod_{p\in\mathbb{P}} p^{y(p)}$.

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. Let us first assume that f is factorable. (i) By the factorability of f

$$\forall\, x \in \mathbb{Z}_+ : f(x, x) = 1.$$

(ii) Let $x, y \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$, where $x = \prod_{p \in \mathbb{P}} p^{x(p)}$ and $y = \prod_{p \in \mathbb{P}} p^{y(p)}$. Let us assume that $y = 1$. Then $x = 1$. Thus $x(p) = 0$ and $y(p) = 0$ for all $p \in \mathbb{P}$, and therefore by (i)

$$f(x, y) = f(1, 1) = \prod_{p \in \mathbb{P}} f(1, 1) = \prod_{p \in \mathbb{P}} f(p^{x(p)}, p^{y(p)}).$$

Let us assume that $y \neq 1$. Then

$$x = \prod_{p \in \mathbb{P}} p^{x(p)} = \prod_{i=1}^{n} p_i^{x(p_i)} \quad \text{and} \quad y = \prod_{p \in \mathbb{P}} p^{y(p)} = \prod_{i=1}^{n} p_i^{y(p_i)},$$

where $n \in \mathbb{Z}_+$ stands for the number of distinct primes in the prime factorization of the element y and $x(p_i) = 0$, $1 \leq i \leq n$, if $p_i \in \mathbb{P}$ is not a prime factor of the element x. Let us use induction on $n \in \mathbb{Z}_+$ to show that

$$\forall\, n \in \mathbb{Z}_+ : f(x, y) = \prod_{i=1}^{n} f(p_i^{x(p_i)}, p_i^{y(p_i)})$$

from which the result follows. If $n = 1$, then

$$f(x, y) = f(\prod_{i=1}^{1} p_i^{x(p_i)}, \prod_{i=1}^{1} p_i^{y(p_i)}) = f(p_1^{x(p_1)}, p_1^{y(p_1)}) = \prod_{i=1}^{1} f(p_i^{x(p_i)}, p_i^{y(p_i)}).$$

Let $n \in \mathbb{Z}_+$, and let us assume that the claim holds for n, that is,

$$f(\prod_{i=1}^{n} p_i^{x(p_i)}, \prod_{i=1}^{n} p_i^{y(p_i)}) = \prod_{i=1}^{n} f(p_i^{x(p_i)}, p_i^{y(p_i)}).$$

Let the number of distinct primes in the prime factorization of the element y be $n + 1$. Then

$$x = \prod_{i=1}^{n+1} p_i^{x(p_i)} \quad \text{and} \quad y = \prod_{i=1}^{n+1} p_i^{y(p_i)}.$$

Since

$$\Big[\prod_{i=1}^{n} p_i^{x(p_i)}\Big] \trianglelefteq \Big[\prod_{i=1}^{n} p_i^{y(p_i)}\Big] \quad \text{and} \quad p_{n+1}^{x(p_{n+1})} \trianglelefteq p_{n+1}^{y(p_{n+1})},$$

and

$$\text{gcf}(\prod_{i=1}^{n} p_i^{x(p_i)}, p_{n+1}^{x(p_{n+1})}) = 1 = \text{gcf}(\prod_{i=1}^{n} p_i^{y(p_i)}, p_{n+1}^{y(p_{n+1})}),$$

and

$$\text{lcm}(\prod_{i=1}^{n} p_i^{x(p_i)}, p_{n+1}^{x(p_{n+1})}) = \prod_{i=1}^{n+1} p_i^{x(p_i)} \quad \text{and} \quad \text{lcm}(\prod_{i=1}^{n} p_i^{y(p_i)}, p_{n+1}^{y(p_{n+1})}) = \prod_{i=1}^{n+1} p_i^{y(p_i)},$$

it follows by the factorability of f and the induction hypothesis that

$$\begin{aligned}
f(x,y) &= f(\prod_{i=1}^{n+1} p_i^{x(p_i)}, \prod_{i=1}^{n+1} p_i^{y(p_i)}) \\
&= f(\operatorname{lcm}(\prod_{i=1}^{n} p_i^{x(p_i)}, p_{n+1}^{x(p_{n+1})}), \operatorname{lcm}(\prod_{i=1}^{n} p_i^{y(p_i)}, p_{n+1}^{y(p_{n+1})})) \\
&= f(\prod_{i=1}^{n} p_i^{x(p_i)}, \prod_{i=1}^{n} p_i^{y(p_i)}) f(p_{n+1}^{x(p_{n+1})}, p_{n+1}^{y(p_{n+1})}) \\
&= \Big[\prod_{i=1}^{n} f(p_i^{x(p_i)}, p_i^{y(p_i)})\Big] f(p_{n+1}^{x(p_{n+1})}, p_{n+1}^{y(p_{n+1})}) \\
&= \prod_{i=1}^{n+1} f(p_i^{x(p_i)}, p_i^{y(p_i)}).
\end{aligned}$$

Thus

$$\forall\, n \in \mathbb{Z}_+ : f(x,y) = \prod_{i=1}^{n} f(p_i^{x(p_i)}, p_i^{y(p_i)}).$$

From the factorability of f it follows that

$$\forall\, p \in \mathbb{P} : x(p) = y(p) = 0 \Rightarrow f(p^{x(p)}, p^{y(p)}) = 1,$$

and therefore

$$f(x,y) = \prod_{p \in \mathbb{P}} f(p^{x(p)}, p^{y(p)}).$$

Thus (i) and (ii) hold.

Let us next assume that (i) and (ii) hold. Then by Theorem 6.4 f is translation invariant and by Theorem 7.4 it is semifactorable, and therefore by Theorem 8.5 f is factorable. □

The following two theorems present sufficient conditions that guarantee the factorability of an arithmetic incidence function.

Theorem 8.10. *If $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is translation invariant and*

(i) $\forall\, x \in \mathbb{Z}_+ : f(x,x) = 1$,

(ii) $\forall\, x, y, z \in \mathbb{Z}_+ : x \trianglelefteq z \trianglelefteq y \Rightarrow f(x,y) = f(x,z)f(z,y)$,

then it is factorable.

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be translation invariant, and let (i) and (ii) hold. Let $x, y, z, w \in \mathbb{Z}_+$ be such that $\operatorname{gcf}(z,w) = x$ and $\operatorname{lcm}(z,w) = y$. Then $x \trianglelefteq z \trianglelefteq y$, and therefore by (ii) $f(x,y) = f(x,z)f(z,y)$. Thus

$$\begin{aligned}
\forall\, x, y, z, w \in \mathbb{Z}_+ : \Big[&\operatorname{gcf}(z,w) = x \text{ and } \operatorname{lcm}(z,w) = y\Big] \\
&\Rightarrow f(x,y) = f(x,z)f(z,y),
\end{aligned}$$

and therefore by Theorem 7.24 f is semifactorable. Since f is translation invariant and semifactorable, it follows by Theorem 8.5 that it is factorable. □

Theorem 8.11. *If* $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ *is such that*

(i) $\forall\, x \in \mathbb{Z}_+ : f(x,x) = 1,$

(ii) $\forall\, x,y,z,w \in \mathbb{Z}_+ : \Big[x \trianglelefteq y \text{ and } z \trianglelefteq w\Big]$
$\Rightarrow f(\mathrm{gcf}(x,z), \mathrm{gcf}(y,w)) f(\mathrm{lcm}(x,z), \mathrm{lcm}(y,w)) = f(x,y) f(z,w),$

then it is factorable.

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be such that (i) and (ii) hold. Let $x,y,z,w \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$, $z \trianglelefteq w$, and $\mathrm{gcf}(x,z) = \mathrm{gcf}(y,w)$. Then by (i) and (ii)

$$\begin{aligned} f(\mathrm{lcm}(x,z), \mathrm{lcm}(y,w)) &= f(\mathrm{gcf}(x,z), \mathrm{gcf}(y,w)) f(\mathrm{lcm}(x,z), \mathrm{lcm}(y,w)) \\ &= f(x,y) f(z,w). \end{aligned}$$

Thus f is factorable. □

Lemma 8.1. *If* $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ *is such that*

$$\begin{aligned} &\forall\, x,y,z,w \in \mathbb{Z}_+ : \Big[x \trianglelefteq y \text{ and } z \trianglelefteq w\Big] \\ &\qquad \Rightarrow f(\mathrm{gcf}(x,z), \mathrm{gcf}(y,w)) f(\mathrm{lcm}(x,z), \mathrm{lcm}(y,w)) = f(x,y) f(z,w), \end{aligned}$$

then

$$\forall\, x,y,z \in \mathbb{Z}_+ : x \trianglelefteq z \trianglelefteq y \Rightarrow f(x,z) f(z,y) = f(x,y) f(z,z).$$

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be such that

$$\begin{aligned} &\forall\, x,y,z,w \in \mathbb{Z}_+ : \Big[x \trianglelefteq y \text{ and } z \trianglelefteq w\Big] \\ &\qquad \Rightarrow f(\mathrm{gcf}(x,z), \mathrm{gcf}(y,w)) f(\mathrm{lcm}(x,z), \mathrm{lcm}(y,w)) = f(x,y) f(z,w). \end{aligned}$$

Let $x,y,z \in \mathbb{Z}_+$ be such that $x \trianglelefteq z \trianglelefteq y$. Then $x \trianglelefteq y$ and $z \trianglelefteq z$, and therefore by the assumption

$$f(\mathrm{gcf}(x,z), \mathrm{gcf}(y,z)) f(\mathrm{lcm}(x,z), \mathrm{lcm}(y,z)) = f(x,y) f(z,z).$$

Thus $f(x,z) f(z,y) = f(x,y) f(z,z)$. □

Lemma 8.2. *A function* $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ *is factorable if and only if*

(i) $\forall\, x \in \mathbb{Z}_+ : f(x,x) = 1,$

(ii) $\forall\, x,y,z,w \in \mathbb{Z}_+ : \Big[x \trianglelefteq y,\ z \trianglelefteq w \text{ and } \mathrm{gcf}(x,z) = \mathrm{gcf}(y,w)\Big]$
$\Rightarrow f(\mathrm{gcf}(x,z), \mathrm{gcf}(y,w)) f(\mathrm{lcm}(x,z), \mathrm{lcm}(y,w)) = f(x,y) f(z,w).$

Proof. Follows by Definition 8.1. □

The conditional statements presented by Theorems 8.10 and 8.11 do not hold conversely. To demonstrate this fact, let $p_1 \in \mathbb{P}$, and let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be such factorable function that

$$f(1, p_1) = f(p_1, p_1^2) = 0 \quad \text{and} \quad f(1, p_1^2) = 1.$$

Then $1 \trianglelefteq p_1 \trianglelefteq p_1^2$ and $f(1, p_1^2) \neq f(1, p_1)f(p_1, p_1^2)$, and therefore

(i) $\neg \forall\, x, y, z \in \mathbb{Z}_+ : x \trianglelefteq z \trianglelefteq y \Rightarrow f(x, y) = f(x, z)f(z, y)$,

(ii) $\neg \forall\, x, y, z \in \mathbb{Z}_+ : x \trianglelefteq z \trianglelefteq y \Rightarrow f(x, z)f(z, y) = f(x, y)f(z, z)$.

From (i) it follows that the conditional statement presented by Theorem 8.10 does not hold conversely. Correspondingly, from (ii) it follows by Lemma 8.1 and the contraposition principle that

$$\begin{aligned}\neg \forall\, x, y, z, w \in \mathbb{Z}_+ : \Big[x \trianglelefteq y \text{ and } z \trianglelefteq w\Big]& \\ \Rightarrow f(\mathrm{gcf}(x, z), \mathrm{gcf}(y, w))f(\mathrm{lcm}(x, z), \mathrm{lcm}(y, w)) &= f(x, y)f(z, w),\end{aligned}$$

which establishes that the conditional statement presented by Theorem 8.11 does not hold conversely, as Lemma 8.2 suggests.

Remark. Theorem 8.11 shows that if $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is such that

$$\forall\, x \in \mathbb{Z}_+ : f(x, x) = 1,$$

and it is factorable in the sense M. Ward [57] defined the concept of factorability of an incidence function, that is,

$$\begin{aligned}\forall\, x, y, z, w \in \mathbb{Z}_+ : \Big[x \trianglelefteq y \text{ and } z \trianglelefteq w\Big]& \\ \Rightarrow f(\mathrm{gcf}(x, z), \mathrm{gcf}(y, w))f(\mathrm{lcm}(x, z), \mathrm{lcm}(y, w)) &= f(x, y)f(z, w),\end{aligned}$$

then it is also factorable in the sense this concept is defined above for arithmetic incidence functions, in other words, it is factorable in the sense D. A. Smith [47] defined the concept of factorability, (see also [47, p. 622]). However, as Lemma 8.2 suggests and the given example establishes, the conditional statement presented by Theorem 8.11 does not hold conversely, contrary to the statement by P. J. McCarthy [25, p. 331]. The inherent difference between these two concepts of factorability is reflected also by the fact that the local distributivity of the underlying local lattice guarantees that the functions satisfying D. A. Smith's factorability condition are closed under the convolution of incidence functions (see Theorem 4.16), whereas the functions satisfying M. Ward's factorability condition are closed under the convolution of incidence functions if and only if the underlying (local) lattice is (locally) Boolean (see [52, p. 237], [4]).

Theorem 8.12. *If $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ are factorable, then $f * g$ is factorable.*

Proof. Let $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be factorable. Then by Theorem 8.5 both f and g are translation invariant and semifactorable, and therefore by Theorem 6.8 $f * g$ is translation invariant and by Theorem 7.7 $f * g$ is semifactorable. Thus by Theorem 8.5 $f * g$ is factorable. □

Theorem 8.13. *If $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is factorable, then f^{*-1} is factorable.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be factorable. Then by Theorem 8.5 f is translation invariant and semifactorable, and therefore by Theorem 6.9 f^{*-1} is translation invariant and by Theorem 7.8 f^{*-1} is semifactorable. Thus by Theorem 8.5 f^{*-1} is factorable. □

Theorem 8.14. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is factorable if and only if its divisibility order summatory function is factorable.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, and let $F \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be the divisibility order summatory function of the function f. Let us first assume that f is factorable. Then by Theorems 8.1 and 8.12 $f * \zeta$ is factorable, and therefore by Theorem 5.15 F is factorable.

Let us next assume that F is factorable. Then by Theorems 8.7 and 8.12 $F * \mu$ is factorable, and therefore by Theorem 5.24 f is factorable. □

Theorem 8.15. *If $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ are factorable, then $f *_C g$ is factorable.*

Proof. Let $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be factorable. Then by Theorem 8.5 both f and g are translation invariant and semifactorable, and therefore by Theorem 6.12 $f *_C g$ is translation invariant and by Theorem 7.11 $f *_C g$ is semifactorable. Thus by Theorem 8.5 $f *_C g$ is factorable. □

Theorem 8.16. *Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be factorable, then f^{*_C-1} is factorable.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be factorable. Then by Theorem 8.5 f is translation invariant and semifactorable, and therefore by Theorem 6.13 f^{*_C-1} is translation invariant and by Theorem 7.12 f^{*_C-1} is semifactorable. Thus by Theorem 8.5 f^{*_C-1} is factorable. □

Theorem 8.17. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is factorable if and only if its complementary summatory function is factorable.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, and let $F \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be the complementary summatory function of the function f. Let us first assume that f is factorable. Then by Theorems 8.1 and 8.15 $f *_C \zeta$ is factorable, and therefore by Theorem 5.33 F is factorable.

Let us next assume that F is factorable. Then by Theorems 8.8 and 8.15 $F *_C \mu_c$ is factorable, and therefore by Theorem 5.42 f is factorable. □

Theorems 8.12, 8.13, and 8.16 and the factorability of the zeta function $\zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, established by Theorem 8.1, offer an alternative method to establish the factorability of the functions $\delta, \mu, \mu_c \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, already established by Theorems 8.6, 8.7, and 8.8.

8.2 Compressibility

As in the case of the semifactorability and the semicompressibility of an arithmetic incidence function, the duality of the concepts of the greatest common factor and the least common multiple suggests also a property that is analogous to the factorability of an arithmetic incidence function. Following the adopted terminology, this property is referred to as compressibility. While building essentially upon the notion of semicompressibility of an arithmetic incidence function, the compressibility introduces a stronger condition for a function to fulfill explaining the chosen terminology.

As a contrast to the relation between the semifactorability and the semicompressibility, it turns out that the property of compressibility is, in fact, a characterization of the factorability of a function, and vice versa.

Definition 8.2. A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is *compressible* if

(i) $\forall\, x \in \mathbb{Z}_+ : f(x,x) = 1$,

(ii) $\forall\, x,y,z,w \in \mathbb{Z}_+ : \Big[x \trianglelefteq y,\ z \trianglelefteq w, \text{ and } \operatorname{lcm}(x,z) = \operatorname{lcm}(y,w)\Big]$
$\Rightarrow f(\operatorname{gcf}(x,z), \operatorname{gcf}(y,w)) = f(x,y)f(z,w)$.

The following two theorems present properties of a compressible function, and therefore, in other words, they give necessary conditions that a function must fulfill in order to be compressible.

Theorem 8.18. *If $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is compressible, then it is translation invariant.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be compressible, and let $x,y,z \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$ and $\operatorname{lcm}(x,z) = \operatorname{lcm}(y,z)$. Since $x \trianglelefteq y$, $z \trianglelefteq z$, and $\operatorname{lcm}(x,z) = \operatorname{lcm}(y,z)$, it follows by the compressibility of f

$$f(\operatorname{gcf}(x,z), \operatorname{gcf}(y,z)) = f(x,y)f(z,z) = f(x,y).$$

Thus by Theorem 6.7 f is translation invariant. $\square$

Theorem 8.19. *If $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is compressible, then it is semicompressible.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be compressible. (i) By the compressibility of f

$$\forall\, x \in \mathbb{Z}_+ : f(x,x) = 1.$$

(ii) Let $x,y,z \in \mathbb{Z}_+$ be such that $y = \operatorname{lcm}(x,z)$. Then $x \trianglelefteq y$, $z \trianglelefteq y$, and $\operatorname{lcm}(x,z) = \operatorname{lcm}(y,y)$, and therefore by the compressibility of f

$$f(\operatorname{gcf}(x,z), y) = f(\operatorname{gcf}(x,z), \operatorname{gcf}(y,y)) = f(x,y)f(z,y).$$

Thus by Definition 7.6 f is semicompressible. $\square$

Lemma 8.3. *Let $x, y, z, w \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$, $z \trianglelefteq w$, and* $\mathrm{lcm}(x, z) = \mathrm{lcm}(y, w)$. *If $u, v \in \mathbb{Z}_+$ are such that $x \trianglelefteq u \trianglelefteq y$ and $z \trianglelefteq v \trianglelefteq w$, then* $\mathrm{lcm}(y, w) = \mathrm{lcm}(u, v)$.

Proof. Elementary. □

The following theorem presents sufficient condition for a function to be compressible, and it captures the effect of combining the properties of translation invariance and semicompressibility.

Theorem 8.20. *If $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is translation invariant and semicompressible, then it is compressible.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be translation invariant and semicompressible. (i) By the semicompressibility of f

$$\forall\, x \in \mathbb{Z}_+ : f(x, x) = 1.$$

(ii) Let $x, y, z, w \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$, $z \trianglelefteq w$, and $\mathrm{lcm}(x, z) = \mathrm{lcm}(y, w)$. Then by Lemma 8.3 $\mathrm{lcm}(x, w) = \mathrm{lcm}(y, w)$ and $\mathrm{lcm}(y, z) = \mathrm{lcm}(y, w)$, and therefore by Theorem 6.7

$$f(x, y) = f(\mathrm{gcf}(x, w), \mathrm{gcf}(y, w)) \quad \text{and} \quad f(z, w) = f(\mathrm{gcf}(y, z), \mathrm{gcf}(y, w)).$$

Since $\mathrm{gcf}(x, w) \trianglelefteq x$, it follows that $\mathrm{gcf}(x, w) \trianglelefteq y$. Thus by Theorems 2.17 and 2.19

$$\begin{aligned}
\mathrm{lcm}(\mathrm{gcf}(x, w), \mathrm{gcf}(y, z)) &= \mathrm{lcm}(\mathrm{gcf}(x, w), \mathrm{gcf}(z, y)) \\
&= \mathrm{gcf}(\mathrm{lcm}(\mathrm{gcf}(x, w), z), y) \\
&= \mathrm{gcf}(\mathrm{lcm}(z, \mathrm{gcf}(x, w)), y) \\
&= \mathrm{gcf}(\mathrm{gcf}(\mathrm{lcm}(z, x), w), y) \\
&= \mathrm{gcf}(y, \mathrm{gcf}(w, \mathrm{lcm}(x, z))) \\
&= \mathrm{gcf}(y, \mathrm{gcf}(w, \mathrm{lcm}(y, w))) \\
&= \mathrm{gcf}(y, \mathrm{gcf}(w, \mathrm{lcm}(w, y))) \\
&= \mathrm{gcf}(y, w),
\end{aligned}$$

and therefore by the semicompressibility of f and Theorem 2.17

$$\begin{aligned}
f(x, y) f(z, w) &= f(\mathrm{gcf}(x, w), \mathrm{gcf}(y, w)) f(\mathrm{gcf}(y, z), \mathrm{gcf}(y, w)) \\
&= f(\mathrm{gcf}(\mathrm{gcf}(x, w), \mathrm{gcf}(y, z)), \mathrm{gcf}(y, w)) \\
&= f(\mathrm{gcf}(\mathrm{gcf}(x, y), \mathrm{gcf}(z, w)), \mathrm{gcf}(y, w)) \\
&= f(\mathrm{gcf}(x, z), \mathrm{gcf}(y, w)).
\end{aligned}$$

By (i) and (ii) f is compressible. □

The following theorem combines the results of the three immediately preceding theorems and presents a characterization of a compressible function, and therefore, in other words, it gives a necessary and sufficient condition for a function to be compressible.

Theorem 8.21. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is compressible if and only if it is translation invariant and semicompressible.*

Proof. Follows by Theorems 8.18, 8.19, and 8.20. $\square$

The following theorem presents another characterization of a compressible function. In effect, this theorem states that the compressibility of a function is a characterization of the factorability of a function, and vice versa.

Theorem 8.22. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is compressible if and only if it is factorable.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. Let us first assume that f is compressible. Then by Theorem 8.21 f is translation invariant and semicompressible, and therefore by Theorem 7.28 it is semifactorable. Thus by Theorem 8.5 f is factorable.

Let us next assume that f is factorable. Then by Theorem 8.5 it is translation invariant and semifactorable, and therefore by Theorem 7.28 it is semicompressible. Thus by Theorem 8.21 f is compressible. $\square$

Remark. If $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is factorable (compressible), then it is both semifactorable and semicompressible, but the converse does not necessarily hold.

Theorem 8.23. *If $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ are compressible, then $f * g$ is compressible.*

Proof. Follows by Theorems 8.22 and 8.12. $\square$

Theorem 8.24. *If $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is compressible, then f^{*-1} is compressible.*

Proof. Follows by Theorems 8.22 and 8.13. $\square$

Theorem 8.25. *If $f, g \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ are compressible, then $f *_C g$ is compressible.*

Proof. Follows by Theorems 8.22 and 8.15. $\square$

Theorem 8.26. *If $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is compressible, then $f^{*_C -1}$ is compressible.*

Proof. Follows by Theorems 8.22 and 8.16. $\square$

8.3 Complete Factorability

The notion of complete factorability of an arithmetic incidence function is, in a sense, another generalization of the notion of complete multiplicativity of an arithmetic function of one variable. While building essentially upon the notion of factorability of an arithmetic incidence function, the complete factorability introduces a stronger condition for a function to fulfill explaining the chosen terminology.

As a contrast to the relation between the factorability of an arithmetic incidence function and the factorability of an incidence function, it turns out that the notion of complete factorability of an arithmetic incidence function is, in fact, a stronger notion than the complete factorability of an incidence function (see Definition 4.16).

Definition 8.3. A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is *completely factorable* if

(i) $\forall\, x \in \mathbb{Z}_+ : f(x,x) = 1$,

(ii) $\forall\, x,y,z,w \in \mathbb{Z}_+ : \Big[x \trianglelefteq y \text{ and } z \trianglelefteq w\Big]$
$$\Rightarrow f(\operatorname{lcm}(x,z), yw/\operatorname{gcf}(x,z)) = f(x,y)f(z,w).$$

Lemma 8.4. *If $x,y,z,w \in \mathbb{Z}_+$ are such that $x \trianglelefteq y$ and $z \trianglelefteq w$, then* $\operatorname{lcm}(x,z) \trianglelefteq yw/\operatorname{gcf}(x,z)$.

Proof. Elementary. □

Theorem 8.27. *The zeta function $\zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely factorable.*

Proof. (i) Let $x \in \mathbb{Z}_+$. Since $x \trianglelefteq x$, it follows that $\zeta(x,x) = 1$. (ii) Let $x,y,z,w \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$ and $z \trianglelefteq w$. Then by Lemma 8.4

$$\zeta(\operatorname{lcm}(x,z), yw/\operatorname{gcf}(x,z)) = 1 = \zeta(x,y)\zeta(z,w).$$

By (i) and (ii) $\zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely factorable. □

The following theorem establishes that if an arithmetic incidence function is completely factorable, then it is necessarily also factorable.

Theorem 8.28. *If $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely factorable, then it is factorable.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be completely factorable. (i) By the complete factorability of f

$$\forall\, x \in \mathbb{Z}_+ : f(x,x) = 1.$$

(ii) Let $x,y,z,w \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$, $z \trianglelefteq w$ and $\operatorname{gcf}(x,z) = \operatorname{gcf}(y,w)$. Then by the complete factorability of f

$$\begin{aligned} f(\operatorname{lcm}(x,z), \operatorname{lcm}(y,w)) &= f(\operatorname{lcm}(x,z), yw/\operatorname{gcf}(y,w)) \\ &= f(\operatorname{lcm}(x,z), yw/\operatorname{gcf}(x,z)) \\ &= f(x,y)f(z,w). \end{aligned}$$

Thus by Definition 8.1 f is factorable. □

The following two theorems present properties of a completely factorable function, and therefore, in other words, they give necessary conditions that a function must fulfill in order to be completely factorable.

Theorem 8.29. *If $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely factorable, then it is completely translation invariant.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be completely factorable, and let $x, y, z \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$. Since $x \trianglelefteq y$ and $xz \trianglelefteq xz$, it follows by the complete factorability of f that

$$f(xz, yz) = f(\operatorname{lcm}(x, xz), yxz/\operatorname{gcf}(x, xz)) = f(x, y)f(xz, xz) = f(x, y).$$

Thus by Definition 6.3 f is completely translation invariant. $\square$

Theorem 8.30. *If $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely factorable, then it is completely semifactorable.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be completely factorable. (i) By the complete factorability of f

$$\forall\, x \in \mathbb{Z}_+ : f(x, x) = 1.$$

(ii) Let $x, y, w \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$ and $x \trianglelefteq w$. Then by the complete factorability of f

$$f(x, yw/x) = f(\operatorname{lcm}(x, x), yw/\operatorname{gcf}(x, x)) = f(x, y)f(x, w).$$

Thus by Definition 7.7 f is completely semifactorable. $\square$

The following theorem presents sufficient condition for a function to be completely factorable, and it captures the effect of combining the properties of complete translation invariance and complete semifactorability.

Theorem 8.31. *If $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely translation invariant and completely semifactorable, then it is completely factorable.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be completely translation invariant and completely semifactorable. (i) By the complete semifactorability of f

$$\forall\, x \in \mathbb{Z}_+ : f(x, x) = 1.$$

(ii) Let $x, y, z, w \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$ and $z \trianglelefteq w$. Then $xz \trianglelefteq yz$ and $xz \trianglelefteq xw$, and therefore by the complete translation invariance and the complete semifactorability of f

$$\begin{aligned}
f(\operatorname{lcm}(x, z), yw/\operatorname{gcf}(x, z)) &= f(\operatorname{gcf}(x, z)\operatorname{lcm}(x, z), \operatorname{gcf}(x, z)(yw/\operatorname{gcf}(x, z))) \\
&= f(xz, yw) \\
&= f(xz, yzxw/xz) \\
&= f(xz, yz)f(xz, xw) \\
&= f(xz, yz)f(zx, wx) \\
&= f(x, y)f(z, w).
\end{aligned}$$

By (i) and (ii) f is completely factorable. $\square$

The following theorem combines the results of the three immediately preceding theorems and presents a characterization of a completely factorable function, and therefore, in other words, it gives a necessary and sufficient condition for a function to be completely factorable.

Theorem 8.32. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely factorable if and only if it is completely translation invariant and completely semifactorable.*

Proof. Follows by Theorems 8.29, 8.30, and 8.31. □

Theorem 8.33. *The delta function $\delta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely factorable.*

Proof. Follows by Theorems 6.19, 7.36, and 8.32. □

Although the notion of complete semifactorability is, in a sense, on a halfway to complete factorability, the notion of factorability seems to be more natural way to approach the complete factorability, since it includes the necessary elements for both the complete translation invariance and the complete semifactorability.

The following theorem presents a prime related property of a completely factorable function, and therefore, in other words, it gives a necessary condition that a function must fulfill in order to be completely factorable.

Theorem 8.34. *If $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely factorable, then*

$$\forall\, x \in \mathbb{Z}_+ : \forall\, p, q \in \mathbb{P} : \forall\, m, n \in \mathbb{N} : f(xq^m, xq^m p^n) = f(x, xp)^n.$$

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be completely factorable, and let $x \in \mathbb{Z}_+$, $p, q \in \mathbb{P}$ and $m \in \mathbb{N}$. Let us use induction on $n \in \mathbb{N}$ to show that

$$\forall\, n \in \mathbb{N} : f(xq^m, xq^m p^n) = f(x, xp)^n$$

from which the result follows. If $n = 0$, then by the complete factorability of f

$$f(xq^m, xq^m p^n) = f(xq^m, xq^m) = 1 = f(x, xp)^n.$$

Let $n \in \mathbb{N}$, and let us assume that $f(xq^m, xq^m p^n) = f(x, xp)^n$. Since $x \trianglelefteq xq^m$, it follows that $\operatorname{gcf}(xq^m, x) = x$ and $\operatorname{lcm}(xq^m, x) = xq^m$. Since $xq^m \trianglelefteq xq^m p^n$ and $x \trianglelefteq xp$, it follows by the complete factorability of f and the induction hypothesis

$$\begin{aligned} f(xq^m, xq^m p^{n+1}) &= f(xq^m, [xq^m p^n \cdot xp]/x) \\ &= f(\operatorname{lcm}(xq^m, x), [xq^m p^n \cdot xp]/\operatorname{gcf}(xq^m, x)) \\ &= f(xq^m, xq^m p^n) f(x, xp) \\ &= f(x, xp)^n f(x, xp) \\ &= f(x, xp)^{n+1}. \end{aligned}$$

Thus

$$\forall\, n \in \mathbb{N} : f(xq^m, xq^m p^n) = f(x, xp)^n.$$

□

The following theorem presents, using primes, a necessary and sufficient condition for a factorable function to be completely factorable.

Theorem 8.35. *Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be factorable. Then f is completely factorable if and only if*

$$\forall\, p \in \mathbb{P} : \forall\, m, n \in \mathbb{N} : f(p^m, p^{m+n}) = f(1, p)^n.$$

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be factorable. Let us first assume that f is completely factorable. Then by Theorem 8.34

$$\forall\, p \in \mathbb{P} : \forall\, m, n \in \mathbb{N} : f(p^m, p^{m+n}) = f(1, p)^n.$$

Let us next assume that

$$\forall\, p \in \mathbb{P} : \forall\, m, n \in \mathbb{N} : f(p^m, p^{m+n}) = f(1, p)^n.$$

(i) By the factorability of f

$$\forall\, x \in \mathbb{Z}_+ : f(x, x) = 1.$$

(ii) Let $x, y, z, w \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$ and $z \trianglelefteq w$. Then $y = xu$ and $w = zv$, where $u, v \in \mathbb{Z}_+$. Thus $yw = xzuv$, and therefore $yw/\operatorname{gcf}(x, z) = \operatorname{lcm}(x, z) \cdot uv$. Thus by Theorem 8.9 and the assumption

$$\begin{aligned}
f(\operatorname{lcm}(x, z), yw/\operatorname{gcf}(x, z)) &= f(\operatorname{lcm}(x, z), \operatorname{lcm}(x, z) \cdot uv) \\
&= \prod_{p \in \mathbb{P}} f(p^{\operatorname{lcm}(x,z)(p)}, p^{[\operatorname{lcm}(x,z) \cdot uv](p)}) \\
&= \prod_{p \in \mathbb{P}} f(p^{\operatorname{lcm}(x,z)(p)}, p^{\operatorname{lcm}(x,z)(p) + (uv)(p)}) \\
&= \prod_{p \in \mathbb{P}} f(1, p)^{(uv)(p)} \\
&= \prod_{p \in \mathbb{P}} f(1, p)^{u(p) + v(p)} \\
&= \prod_{p \in \mathbb{P}} f(1, p)^{u(p)} f(1, p)^{v(p)} \\
&= \prod_{p \in \mathbb{P}} f(p^{x(p)}, p^{x(p) + u(p)}) f(p^{z(p)}, p^{z(p) + v(p)}) \\
&= \prod_{p \in \mathbb{P}} f(p^{x(p)}, p^{(xu)(p)}) f(p^{z(p)}, p^{(zv)(p)}) \\
&= \prod_{p \in \mathbb{P}} f(p^{x(p)}, p^{y(p)}) f(p^{z(p)}, p^{w(p)}) \\
&= \Big[\prod_{p \in \mathbb{P}} f(p^{x(p)}, p^{y(p)})\Big] \Big[\prod_{p \in \mathbb{P}} f(p^{z(p)}, p^{w(p)})\Big] \\
&= f(x, y) f(z, w).
\end{aligned}$$

By (i) and (ii) f is completely factorable. $\square$

The following theorem presents a prime related characterization of a completely factorable function.

Theorem 8.36. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely factorable if and only if*

(i) $\forall\, x \in \mathbb{Z}_+ : f(x, x) = 1$,

(ii) $\forall\, x, y \in \mathbb{Z}_+ : x \trianglelefteq y \Rightarrow f(x, y) = \prod_{p\in\mathbb{P}} f(1, p)^{y(p)-x(p)}$,
where $x = \prod_{p\in\mathbb{P}} p^{x(p)}$ *and* $y = \prod_{p\in\mathbb{P}} p^{y(p)}$.

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. Let us first assume that f is completely factorable.
(i) By the complete factorability of f

$$\forall\, x \in \mathbb{Z}_+ : f(x, x) = 1.$$

(ii) Let $x, y \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$, where $x = \prod_{p\in\mathbb{P}} p^{x(p)}$ and $y = \prod_{p\in\mathbb{P}} p^{y(p)}$.
Then by Theorems 8.28, 8.9, and 8.35

$$f(x, y) = \prod_{p\in\mathbb{P}} f(p^{x(p)}, p^{y(p)}) = \prod_{p\in\mathbb{P}} f(p^{x(p)}, p^{x(p)+y(p)-x(p)}) = \prod_{p\in\mathbb{P}} f(1, p)^{y(p)-x(p)}.$$

Thus (i) and (ii) hold.

Let us next assume that (i) and (ii) hold. Let $x, y, z, w \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$ and $z \trianglelefteq w$, where $x = \prod_{p\in\mathbb{P}} p^{x(p)}$, $y = \prod_{p\in\mathbb{P}} p^{y(p)}$, $z = \prod_{p\in\mathbb{P}} p^{z(p)}$, and $w = \prod_{p\in\mathbb{P}} p^{w(p)}$. Then $y = xu$ and $w = zv$, where $u, v \in \mathbb{Z}_+$. Thus $yw = xzuv$, and therefore $yw/\gcf(x, z) = \operatorname{lcm}(x, z) \cdot uv$. Thus by (ii)

$$\begin{aligned}
f(\operatorname{lcm}(x, z), yw/\operatorname{gcf}(x, z)) &= f(\operatorname{lcm}(x, z), \operatorname{lcm}(x, z) \cdot uv) \\
&= \prod_{p\in\mathbb{P}} f(1, p)^{[\operatorname{lcm}(x,z)\cdot uv](p)-\operatorname{lcm}(x,z)(p)} \\
&= \prod_{p\in\mathbb{P}} f(1, p)^{\operatorname{lcm}(x,z)(p)+(uv)(p)-\operatorname{lcm}(x,z)(p)} \\
&= \prod_{p\in\mathbb{P}} f(1, p)^{u(p)+v(p)} \\
&= \prod_{p\in\mathbb{P}} f(1, p)^{u(p)} f(1, p)^{v(p)} \\
&= \prod_{p\in\mathbb{P}} f(1, p)^{y(p)-x(p)} f(1, p)^{w(p)-z(p)} \\
&= \Big[\prod_{p\in\mathbb{P}} f(1, p)^{y(p)-x(p)}\Big]\Big[\prod_{p\in\mathbb{P}} f(1, p)^{w(p)-z(p)}\Big] \\
&= f(x, y) f(z, w).
\end{aligned}$$

Thus f is completely factorable. $\square$

Theorem 8.37. *If $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely factorable then*

$$\forall\, x, y, z, w \in \mathbb{Z}_+ : \Big[x \trianglelefteq y \text{ and } z \trianglelefteq w\Big] \\ \Rightarrow f(\operatorname{gcf}(x, z), \operatorname{gcf}(y, w)) f(\operatorname{lcm}(x, z), \operatorname{lcm}(y, w)) = f(x, y) f(z, w).$$

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be completely factorable, and let $x, y, z, w \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$ and $z \trianglelefteq w$. Then $y = xu$ and $w = zv$, where $u, v \in \mathbb{Z}_+$, and therefore $yw = xzuv$. Since $x \trianglelefteq y$ and $z \trianglelefteq w$, it follows that $\operatorname{gcf}(x, z) \trianglelefteq \operatorname{gcf}(y, w)$ and $\operatorname{lcm}(x, z) \trianglelefteq \operatorname{lcm}(y, w)$, and therefore $\operatorname{gcf}(y, w) = \operatorname{gcf}(x, z) \cdot s$ and $\operatorname{lcm}(y, w) = \operatorname{lcm}(x, z) \cdot t$, where $s, t \in \mathbb{Z}_+$. Thus $\operatorname{gcf}(y, w) \operatorname{lcm}(y, w) = \operatorname{gcf}(x, z) \operatorname{lcm}(x, z) \cdot st$, and therefore $yw = xzst$. Since $yw = xzuv$ and $yw = xzst$, it follows that $uv = st$. Thus by Theorems 8.28, 8.9 and 8.35

$$\begin{aligned}
&f(\operatorname{gcf}(x, z), \operatorname{gcf}(y, w)) f(\operatorname{lcm}(x, z), \operatorname{lcm}(y, w)) \\
&\quad = f(\operatorname{gcf}(x, z), \operatorname{gcf}(x, z) \cdot s) f(\operatorname{lcm}(x, z), \operatorname{lcm}(x, z) \cdot t) \\
&\quad = \Big[\prod_{p \in \mathbb{P}} f(p^{\operatorname{gcf}(x,z)(p)}, p^{[\operatorname{gcf}(x,z)\cdot s](p)})\Big]\Big[\prod_{p \in \mathbb{P}} f(p^{\operatorname{lcm}(x,z)(p)}, p^{[\operatorname{lcm}(x,z)\cdot t](p)})\Big] \\
&\quad = \Big[\prod_{p \in \mathbb{P}} f(p^{\operatorname{gcf}(x,z)(p)}, p^{\operatorname{gcf}(x,z)(p)+s(p)})\Big]\Big[\prod_{p \in \mathbb{P}} f(p^{\operatorname{lcm}(x,z)(p)}, p^{\operatorname{lcm}(x,z)(p)+t(p)})\Big] \\
&\quad = \Big[\prod_{p \in \mathbb{P}} f(1, p)^{s(p)}\Big]\Big[\prod_{p \in \mathbb{P}} f(1, p)^{t(p)}\Big] \\
&\quad = \prod_{p \in \mathbb{P}} f(1, p)^{s(p)} f(1, p)^{t(p)} \\
&\quad = \prod_{p \in \mathbb{P}} f(1, p)^{s(p)+t(p)} \\
&\quad = \prod_{p \in \mathbb{P}} f(1, p)^{(st)(p)} \\
&\quad = \prod_{p \in \mathbb{P}} f(1, p)^{(uv)(p)} \\
&\quad = \prod_{p \in \mathbb{P}} f(1, p)^{u(p)+v(p)} \\
&\quad = \prod_{p \in \mathbb{P}} f(1, p)^{u(p)} f(1, p)^{v(p)} \\
&\quad = \prod_{p \in \mathbb{P}} f(p^{x(p)}, p^{x(p)+u(p)}) f(p^{z(p)}, p^{z(p)+v(p)}) \\
&\quad = \prod_{p \in \mathbb{P}} f(p^{x(p)}, p^{(xu)(p)}) f(p^{z(p)}, p^{(zv)(p)}) \\
&\quad = \prod_{p \in \mathbb{P}} f(p^{x(p)}, p^{y(p)}) f(p^{z(p)}, p^{w(p)}) \\
&\quad = \Big[\prod_{p \in \mathbb{P}} f(p^{x(p)}, p^{y(p)})\Big]\Big[\prod_{p \in \mathbb{P}} f(p^{z(p)}, p^{w(p)})\Big] \\
&\quad = f(x, y) f(z, w).
\end{aligned}$$

□

The following two theorems present characterizations of a completely factorable function.

Theorem 8.38. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely factorable if and only if it is completely translation invariant and*

(i) $\forall\, x \in \mathbb{Z}_+ : f(x, x) = 1,$

(ii) $\forall\, x, y, z \in \mathbb{Z}_+ : x \trianglelefteq z \trianglelefteq y \Rightarrow f(x, y) = f(x, z)f(z, y).$

Proof. Follows by Theorems 8.32 and 7.51. □

Theorem 8.39. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely factorable if and only if it is completely translation invariant and*

(i) $\forall\, x \in \mathbb{Z}_+ : f(x, x) = 1,$

(ii) $\forall\, x, y, z, w \in \mathbb{Z}_+ : \Big[x \trianglelefteq y \text{ and } z \trianglelefteq w\Big]$
$\Rightarrow f(\gcd(x, z), \gcd(y, w))f(\operatorname{lcm}(x, z), \operatorname{lcm}(y, w)) = f(x, y)f(z, w).$

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. Let us first assume that f is completely factorable. Then by Theorem 8.32 f is completely translation invariant, (i) holds by the complete factorability of f, and (ii) holds by Theorem 8.37.

Let us next assume that f is completely translation invariant and that (i) and (ii) hold. From (ii) it follows by Lemma 8.1 that

$$\forall\, x, y, z \in \mathbb{Z}_+ : x \trianglelefteq z \trianglelefteq y \Rightarrow f(x, z)f(z, y) = f(x, y)f(z, z),$$

and therefore by (i)

$$\forall\, x, y, z \in \mathbb{Z}_+ : x \trianglelefteq z \trianglelefteq y \Rightarrow f(x, y) = f(x, z)f(z, y).$$

Thus by Theorem 8.38 f is completely factorable. □

Theorem 8.38 establishes, essentially, that the complete factorability of an arithmetic incidence function (see Definition 8.3) is a stronger notion than the complete factorability of an incidence function (see Definition 4.16). In order to demonstrate this fact, let $p_1, p_2 \in \mathbb{P}$ be distinct, and let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be a factorable function such that

$$\begin{array}{llll}
f(1, p_1) = p_1, & f(1, p_1p_2) = p_1p_2 & f(p_1, p_1p_2) = p_2, & f(p_2, p_1^2p_2) = p_1^3, \\
f(1, p_2) = p_2, & f(1, p_1^2p_2) = p_1^3p_2, & f(p_1, p_1^2p_2) = p_1^2p_2, & f(p_1^2, p_1^2p_2) = p_2, \\
f(1, p_1^2) = p_1^3 & f(p_1, p_1^2) = p_1^2, & f(p_2, p_1p_2) = p_1, & f(p_1p_2, p_1^2p_2) = p_1^2.
\end{array}$$

Since f is factorable (see Definition 8.1), it follows by Theorem 8.5 that it is translation invariant (see Definition 6.2). It can also be verified that

$$\forall\, x, y, z \in \mathbb{Z}_+ : x \trianglelefteq z \trianglelefteq y \Rightarrow f(x, y) = f(x, z)f(z, y),$$

and therefore f is completely factorable in the sense this notion is defined for incidence functions (see Definition 4.16). However, although f is translation invariant, it is not completely translation invariant (e.g. $f(p_1, p_1^2) \neq f(1, p_1)$), and therefore it is not completely factorable in the sense this notion is defined above for arithmetic incidence functions (see Definition 8.3).

Remark. Theorem 8.10 establishes that if $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely factorable in the sense this notion is defined for incidence functions (see Definition 4.16), then it is also factorable (see Definition 8.1). On the other hand, Theorem 8.11 establishes that if $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is such that

$$\forall\, x \in \mathbb{Z}_+ : f(x, x) = 1,$$

and it is factorable in the sense M. Ward [57] defined the concept of factorability of an incidence function, that is,

$$\begin{aligned}\forall\, x, y, z, w \in \mathbb{Z}_+ : &\Big[x \trianglelefteq y \text{ and } z \trianglelefteq w\Big] \\ &\Rightarrow f(\mathrm{gcf}(x, z), \mathrm{gcf}(y, w)) f(\mathrm{lcm}(x, z), \mathrm{lcm}(y, w)) = f(x, y) f(z, w),\end{aligned}$$

then it is factorable (see Definition 8.1). Since, in contrast to Definition 4.16, the translation invariance of a function is not explicitly included in the hypothesis of Theorem 8.11, this brings up the question whether M. Ward's concept of factorability of an incidence function would be, in fact, a more appropriate way to define the complete factorability of an incidence function. This strategy is backed up by Lemma 8.1 which, albeit stated specifically to concern the arithmetic incidence functions, establishes that this 'suggested alternative definition' of a complete factorability of an incidence function would be at least as strong as the present definition (Definition 4.16). Moreover, Theorem 8.39 establishes, essentially, that the complete factorability of an arithmetic incidence function (see Definition 8.3) is a stronger notion than the complete factorability of an incidence function suggested above. However, in contrast to the present definitions of a complete factorability (Definition 4.16) and a complete factorability of an arithmetic incidence function (Definition 8.3), the suggested alternative definition, applying M. Ward's factorability condition, 'replaces two intervals with two intervals' instead of 'replacing one interval with two intervals', which in this case can be regarded as a somewhat unsatisfactory feature. On the other hand, this situation is analogous to that of Theorem 7.6, which presents an alternative way to define the semifactorability of an arithmetic incidence function (see Definition 7.1) and consequently the 'semifactorability of an incidence function'.

Since the D-convolution and the C-convolution do not necessarily preserve the complete semifactorability, it follows by Theorem 8.32 that they do not necessarily preserve the complete factorability. Correspondingly, since neither the D-convolution inverse nor the C-convolution inverse of a completely semifactorable function is necessarily completely semifactorable, it follows by Theorem 8.32 that neither of them need be completely factorable.

8.4 Complete Compressibility

The evident symmetry that holds between the factorability and the compressibility of an arithmetic incidence function and the duality of the concepts of the greatest common factor and the least common multiple suggest also a property that is analogous to the complete factorability of an arithmetic incidence function. Following the adopted terminology, this property is referred to as complete compressibility. While building essentially upon the notion of compressibility of an arithmetic incidence function, the complete compressibility introduces a stronger condition for a function to fulfill explaining the chosen terminology.

As in the case of factorability and compressibility, it turns out that the property of complete compressibility is, in effect, a characterization of the complete factorability of a function, and vice versa.

Definition 8.4. A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is *completely compressible* if

(i) $\forall\, x \in \mathbb{Z}_+ : f(x,x) = 1$,

(ii) $\forall\, x,y,z,w \in \mathbb{Z}_+ : \Big[x \trianglelefteq y,\ z \trianglelefteq w,\ \text{and } \operatorname{lcm}(y,w) \trianglelefteq xz\Big]$
$\Rightarrow f(xz/\operatorname{lcm}(y,w), \operatorname{gcf}(y,w)) = f(x,y)f(z,w)$.

Lemma 8.5. *If* $x,y,z,w \in \mathbb{Z}_+$ *are such that* $x \trianglelefteq y$, $z \trianglelefteq w$, *and* $\operatorname{lcm}(y,w) \trianglelefteq xz$, *then* $xz/\operatorname{lcm}(y,w) \trianglelefteq x$ *and* $xz/\operatorname{lcm}(y,w) \trianglelefteq z$.

Proof. Elementary. □

Remark. If $x,y,z,w \in \mathbb{Z}_+$ are such that $x \trianglelefteq y$, $z \trianglelefteq w$, and $\operatorname{lcm}(y,w) \trianglelefteq xz$, then by Lemma 8.5 and Theorem 2.14 $xz/\operatorname{lcm}(y,w) \trianglelefteq \operatorname{gcf}(x,z)$, and therefore $xz/\operatorname{lcm}(y,w) \trianglelefteq \operatorname{gcf}(y,w)$.

The following theorem establishes that if an arithmetic incidence function is completely compressible, then it is necessarily also compressible.

Theorem 8.40. *If* $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ *is completely compressible, then it is compressible.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be completely compressible. (i) By the complete compressibility of f

$$\forall\, x \in \mathbb{Z}_+ : f(x,x) = 1.$$

(ii) Let $x,y,z,w \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$, $z \trianglelefteq w$ and $\operatorname{lcm}(x,z) = \operatorname{lcm}(y,w)$. Since $\operatorname{gcf}(x,z)\operatorname{lcm}(x,z) = xz$, it follows that $\operatorname{lcm}(y,w) \trianglelefteq xz$, and therefore by the complete compressibility of f

$$\begin{aligned} f(\operatorname{gcf}(x,z), \operatorname{gcf}(y,w)) &= f(xz/\operatorname{lcm}(x,z), \operatorname{gcf}(y,w)) \\ &= f(xz/\operatorname{lcm}(y,w), \operatorname{gcf}(y,w)) \\ &= f(x,y)f(z,w). \end{aligned}$$

Thus by Definition 8.2 f is compressible. □

The following two theorems present properties of a completely compressible function, and therefore, in other words, they give necessary conditions that a function must fulfill in order to be completely compressible.

Theorem 8.41. *If $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely compressible, then it is completely translation invariant.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be completely compressible, and let $x, y, z \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$. Since $y \trianglelefteq y$, $xz \trianglelefteq yz$, and $\operatorname{lcm}(y, yz) \trianglelefteq yxz$, it follows by the complete compressibility of f that

$$f(x, y) = f(yxz/\operatorname{lcm}(y, yz), \operatorname{gcf}(y, yz)) = f(y, y)f(xz, yz) = f(xz, yz).$$

Thus by Definition 6.3 f is completely translation invariant. □

Theorem 8.42. *If $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely compressible, then it is completely semicompressible.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be completely compressible. (i) By the complete compressibility of f

$$\forall\, x \in \mathbb{Z}_+ : f(x, x) = 1.$$

(ii) Let $x, y, z \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$, $z \trianglelefteq y$, and $y \trianglelefteq xz$. Then $\operatorname{lcm}(y, y) \trianglelefteq xz$, and therefore by the complete compressibility of f

$$f(xz/y, y) = f(xz/\operatorname{lcm}(y, y), \operatorname{gcf}(y, y)) = f(x, y)f(z, y).$$

Thus by Definition 7.10 f is completely semicompressible. □

The following theorem presents sufficient conditions for a function to be completely compressible, and it captures the effect of combining the properties of complete translation invariance and complete semicompressibility.

Theorem 8.43. *If $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely translation invariant and completely semicompressible, then it is completely compressible.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ be completely translation invariant and completely semicompressible. (i) By the complete semicompressibility of f

$$\forall\, x \in \mathbb{Z}_+ : f(x, x) = 1.$$

(ii) Let $x, y, z, w \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$ and $z \trianglelefteq w$. Then $xw \trianglelefteq yw$, $yz \trianglelefteq yw$, and $yw \trianglelefteq xw \cdot yz$, and therefore by the complete translation invariance and the complete semicompressibility of f

$$\begin{aligned} f(xz/\operatorname{lcm}(y, w), \operatorname{gcf}(y, w)) &= f((xz/\operatorname{lcm}(y, w))\operatorname{lcm}(y, w), \operatorname{gcf}(y, w)\operatorname{lcm}(y, w)) \\ &= f(xz, yw) \\ &= f(xwyz/yw, yw) \\ &= f(xw, yw)f(yz, yw) \\ &= f(xw, yw)f(zy, wy) \\ &= f(x, y)f(z, w). \end{aligned}$$

By (i) and (ii) f is completely compressible. □

The following theorem combines the results of the three immediately preceding theorems and presents a characterization of a completely compressible function, and therefore, in other words, it gives a necessary and sufficient condition for a function to be completely compressible.

Theorem 8.44. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely compressible if and only if it is completely translation invariant and completely semicompressible.*

Proof. Follows by Theorems 8.41, 8.42, and 8.43. $\square$

The following theorem presents another characterization of a completely compressible function. In effect, this theorem states that the complete compressibility of a function is a characterization of the complete factorability of a function, and vice versa.

Theorem 8.45. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely compressible if and only if it is completely factorable.*

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. Let us first assume that f is completely compressible. Then by Theorem 8.44 f is completely translation invariant and completely semicompressible, and therefore by Theorem 7.57 it is completely semifactorable. Thus by Theorem 8.32 f is completely factorable.

Let us next assume that f is completely factorable. Then by Theorem 8.32 it is completely translation invariant and completely semifactorable, and therefore by Theorem 7.57 it is completely semicompressible. Thus by Theorem 8.44 f is completely compressible. $\square$

Remark. If $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely factorable (completely compressible), then it is both completely semifactorable and completely semicompressible, but the converse does not necessarily hold.

Since the D-convolution and the C-convolution do not necessarily preserve the complete semicompressibility, it follows by Theorem 8.44 that they do not necessarily preserve the complete compressibility. Correspondingly, since neither the D-convolution inverse nor the C-convolution inverse of a completely semicompressible function is necessarily completely semicompressible, it follows by Theorem 8.44 that neither of them need be completely compressible.

Chapter 9

Some Special Functions

9.1 Prime Factor Product Function

The *prime factor product function* of the factor lattice $(\mathbb{Z}_+, \trianglelefteq)$ is a generalization of the corresponding arithmetic function. This function has already been introduced earlier (see Definition 5.5), but let us present it once again.

Definition 9.1. The prime factor product function $\gamma \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is defined as follows:

$$\gamma : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C} : \gamma(x, y) = \begin{cases} \prod\limits_{\substack{px \trianglelefteq y \\ p \in \mathbb{P}}} p, & \text{if } x \trianglelefteq y; \\ 0, & \text{otherwise,} \end{cases}$$

i.e. the value of $\gamma(x, y)$ is the product of distinct prime factors of y/x.

Theorem 9.1. *The prime factor product function $\gamma \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely translation invariant.*

Proof. Let $x, y, z \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$. Then

$$\gamma(xz, yz) = \prod_{\substack{pxz \trianglelefteq yz \\ p \in \mathbb{P}}} p = \prod_{\substack{px \trianglelefteq y \\ p \in \mathbb{P}}} p = \gamma(x, y).$$

Thus by Definition 6.3 $\gamma \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely translation invariant. $\square$

Theorem 9.2. *The prime factor product function $\gamma \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is semifactorable.*

Proof. (i) Let $x \in \mathbb{Z}_+$. Then

$$\gamma(x, x) = \prod_{\substack{px \trianglelefteq x \\ p \in \mathbb{P}}} p = \prod_{\substack{p \trianglelefteq 1 \\ p \in \mathbb{P}}} p = 1.$$

(ii) Let $x, y, w \in \mathbb{Z}_+$ be such that $x = \mathrm{gcf}(y, w)$, where $x = \prod_{p\in\mathbb{P}} p^{x(p)}$, $y = \prod_{p\in\mathbb{P}} p^{y(p)}$, and $w = \prod_{p\in\mathbb{P}} p^{w(p)}$. Let us assume that $p \in \mathbb{P}$ is such that $px \trianglelefteq \mathrm{lcm}(y, w)$. Then $(px)(p) \leq \mathrm{lcm}(y, w)(p)$, and therefore

$$x(p) + 1 \leq \max\{y(p), w(p)\}.$$

Let us note that since $x \trianglelefteq y$ and $x \trianglelefteq w$, and $\mathrm{gcf}(x, x) = \mathrm{gcf}(y, w)$, it follows by Lemma 6.3 that

$$\Big[\max\{y(p), w(p)\} = y(p) \text{ and } x(p) = w(p)\Big] \text{ or }$$
$$\Big[\max\{y(p), w(p)\} = w(p) \text{ and } x(p) = y(p)\Big].$$

Let us specifically note that if $\max\{y(p), w(p)\} = y(p)$ and $x(p) = w(p)$, then

$$x(p) + 1 \leq y(p) \quad \text{and} \quad x(p) + 1 \not\leq w(p).$$

Correspondingly, if $\max\{y(p), w(p)\} = w(p)$ and $x(p) = y(p)$, then

$$x(p) + 1 \not\leq y(p) \quad \text{and} \quad x(p) + 1 \leq w(p).$$

Thus

$$\Big[(px)(p) \leq y(p) \text{ and } (px)(p) \not\leq w(p)\Big] \text{ or } \Big[(px)(p) \not\leq y(p) \text{ and } (px)(p) \leq w(p)\Big],$$

and therefore

$$\Big[px \trianglelefteq y \text{ and } px \ntrianglelefteq w\Big] \text{ or } \Big[px \ntrianglelefteq y \text{ and } px \trianglelefteq w\Big].$$

Thus

$$\forall\, p \in \mathbb{P} : px \trianglelefteq \mathrm{lcm}(y, w) \Rightarrow \Big[\Big[px \trianglelefteq y \text{ and } px \ntrianglelefteq w\Big] \text{ or } \Big[px \ntrianglelefteq y \text{ and } px \trianglelefteq w\Big]\Big],$$

and therefore

$$\gamma(x, \mathrm{lcm}(y, w)) = \prod_{\substack{px \trianglelefteq \mathrm{lcm}(y,w) \\ p\in\mathbb{P}}} p = \Big[\prod_{\substack{px \trianglelefteq y \\ p\in\mathbb{P}}} p\Big]\Big[\prod_{\substack{px \trianglelefteq w \\ p\in\mathbb{P}}} p\Big] = \gamma(x, y)\gamma(x, w).$$

Thus by Definition 7.1 $\gamma \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is semifactorable. □

Theorem 9.3. *The prime factor product function $\gamma \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is factorable.*

Proof. By Theorems 9.1 and 6.17 $\gamma \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is translation invariant, and by Theorem 9.2 it is semifactorable, and therefore by Theorem 8.5 it is factorable. □

9.2 Power Functions

The family of *power functions* of the factor lattice $(\mathbb{Z}_+, \trianglelefteq)$ is a generalization of the corresponding arithmetic functions.

Definition 9.2. Let $\alpha \in \mathbb{N}$. The *power function* $\zeta_\alpha \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is defined as follows:

$$\zeta_\alpha : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C} : \zeta_\alpha(x, y) = \begin{cases} (y/x)^\alpha, & \text{if } x \trianglelefteq y; \\ 0, & \text{otherwise.} \end{cases}$$

If $x, y \in \mathbb{Z}_+$ are such that $x \trianglelefteq y$, then $\zeta_0(x, y) = (y/x)^0 = 1$, and therefore the power function $\zeta_0 \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is the same function as the zeta function $\zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$.

Theorem 9.4. *The power functions $\zeta_\alpha \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ are completely translation invariant.*

Proof. Let $\alpha \in \mathbb{N}$, and let $x, y, z \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$. Then $yz/xz = y/x$, and therefore

$$\zeta_\alpha(xz, yz) = (yz/xz)^\alpha = (y/x)^\alpha = \zeta_\alpha(x, y).$$

Thus by Definition 6.3 $\zeta_\alpha \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely translation invariant. $\square$

Theorem 9.5. *The power functions $\zeta_\alpha \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ are completely semifactorable.*

Proof. Let $\alpha \in \mathbb{N}$. (i) Let $x \in \mathbb{Z}_+$. Then

$$\zeta_\alpha(x, x) = (x/x)^\alpha = 1^\alpha = 1.$$

(ii) Let $x, y, w \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$ and $x \trianglelefteq w$. Then $y = xu$ and $w = xv$, where $u, v \in \mathbb{Z}_+$. Thus $yw = xuxv$, and therefore $yw/x = xuv$. Thus $(yw/x)/x = uv$, and therefore $(yw/x)/x = y/x \cdot w/x$. Thus

$$\begin{aligned} \zeta_\alpha(x, yw/x) &= ((yw/x)/x)^\alpha \\ &= (y/x \cdot w/x)^\alpha \\ &= (y/x)^\alpha (w/x)^\alpha \\ &= \zeta_\alpha(x, y)\zeta_\alpha(x, w). \end{aligned}$$

Thus by Definition 7.7 $\zeta_\alpha \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely semifactorable. $\square$

Theorem 9.6. *The power functions $\zeta_\alpha \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ are completely factorable.*

Proof. Follows by Theorems 9.4, 9.5, and 8.32. $\square$

9.3 The Euler Function

The *Euler function* of the factor lattice $(\mathbb{Z}_+, \trianglelefteq)$ is a generalization of the corresponding arithmetic function.

Definition 9.3. Let $x, y \in \mathbb{Z}_+$. The *Euler set* $\phi_{x,y}$ is defined as follows:

$$\phi_{x,y} = \{z \mid z \in \mathbb{Z}_+,\ x \leq z \leq y, \text{ and } \operatorname{gcf}(z, y) = x\}.$$

Lemma 9.1. *Let $x, y \in \mathbb{Z}_+$. Then $\phi_{x,y} \neq \emptyset$ if and only if $x \trianglelefteq y$.*

Proof. Let $x, y \in \mathbb{Z}_+$. Let us first assume that $\phi_{x,y} \neq \emptyset$. Then there is $z \in \mathbb{Z}_+$ such that $x \leq z \leq y$ and $\operatorname{gcf}(z, y) = x$. Thus $x \trianglelefteq y$. Let us next assume that $x \trianglelefteq y$. Since $x \in \mathbb{Z}_+$, $x \leq x \leq y$, and $\operatorname{gcf}(x, y) = x$, it follows that $x \in \phi_{x,y}$, and therefore $\phi_{x,y} \neq \emptyset$. □

Definition 9.4. The *Euler function* $\phi \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is defined as follows:

$$\phi : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C} : \phi(x, y) = \#\phi_{x,y},$$

where $\#\phi_{x,y}$ denotes the number of the elements of the Euler set $\phi_{x,y}$.

Theorem 9.7. *The Euler function $\phi \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely translation invariant.*

Proof. Let $x, y \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$. Let us show that $\#\phi_{x,y} = \#\phi_{1,y/x}$ from which it follows that $\phi(x, y) = \phi(1, y/x)$. (i) If $z \in \phi_{x,y}$, then $z \in \mathbb{Z}_+$, $x \leq z \leq y$, and $\operatorname{gcf}(z, y) = x$. Thus $x \trianglelefteq z$, and therefore $z/x \in \mathbb{Z}_+$. Since $1 \leq z/x \leq y/x$ and $\operatorname{gcf}(z/x, y/x) = 1$, it follows that $z/x \in \phi_{1,y/x}$. (ii) If $z/x \in \phi_{1,y/x}$, then $z/x \in \mathbb{Z}_+$, $1 \leq z/x \leq y/x$, and $\operatorname{gcf}(z/x, y/x) = 1$. Since $x, z/x \in \mathbb{Z}_+$ and $x \cdot z/x = z$, it follows that $z \in \mathbb{Z}_+$. Since $x \leq z \leq y$ and $\operatorname{gcf}(z, y) = x$, it follows that $z \in \phi_{x,y}$. From (i) and (ii) it follows that $z \in \phi_{x,y}$ if and only if $z/x \in \phi_{1,y/x}$. Thus $\#\phi_{x,y} = \#\phi_{1,y/x}$, and therefore $\phi(x, y) = \phi(1, y/x)$. Thus by Theorem 6.18 $\phi \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely translation invariant. □

Lemma 9.2. *The following holds for the Euler function $\phi \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$:*

$$\forall\, x, y \in \mathbb{Z}_+ : x \trianglelefteq y \Rightarrow y/x = \sum_{x \trianglelefteq z \trianglelefteq y} \phi(x, z),$$

Proof. Let $x, y \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$. Then $y/x \in \mathbb{Z}_+$. If $z \in \mathbb{Z}_+$ is such that $z \trianglelefteq y/x$, then $(y/x)/z \in \mathbb{Z}_+$. Let us define the set S_z, where $z \in \mathbb{Z}_+$ is such that $z \trianglelefteq y/x$, as follows:

$$S_z = \{u((y/x)/z) \mid u \in \phi_{1,z}\}.$$

Since $\phi_{1,z} \neq \emptyset$, it follows that $S_z \neq \emptyset$. If $u \in \phi_{1,z}$, then $u \in \mathbb{Z}_+$, and therefore $u((y/x)/z) \in \mathbb{Z}_+$. From the definition of S_z it follows that $\#S_z = \#\phi_{1,z}$. Let us show that

$$\{s \mid s \in \mathbb{Z}_+,\, s \leq y/x\} = \bigcup_{1 \trianglelefteq z \trianglelefteq y/x} S_z$$

and

$$\forall\, z, w \in \mathbb{Z}_+ : (z \trianglelefteq y/x,\, w \trianglelefteq y/x,\, z \neq w) \Rightarrow S_z \cap S_w = \emptyset,$$

from which the result follows. (i) Let $t \in \{s \mid s \in \mathbb{Z}_+,\, s \leq y/x\}$. Then $y/x = z \cdot \mathrm{gcf}(t, y/x)$, where $z \in \mathbb{Z}_+$, and therefore $z \trianglelefteq y/x$. Let us show that $t \in S_z$. Since $\mathrm{gcf}(t, y/x) = (y/x)/z$, it follows that $(y/x)/z \trianglelefteq t$, and therefore $t = u((y/x)/z)$, where $u \in \mathbb{Z}_+$. Thus

$$\begin{aligned} & t \leq y/x \\ \Leftrightarrow\quad & u((y/x)/z) \leq y/x \\ \Leftrightarrow\quad & u((y/x)/z)z \leq (y/x)z \\ \Leftrightarrow\quad & u(y/x) \leq (y/x)z \\ \Leftrightarrow\quad & u \leq z. \end{aligned}$$

Since $u \in \mathbb{Z}_+$, it follows that $1 \leq u \leq z$. Since $\mathrm{gcf}(t, y/x) = (y/x)/z$, it follows that $\mathrm{gcf}(u((y/x)/z), y/x) = (y/x)/z$, and therefore $\mathrm{gcf}(u, z) = 1$. Thus $u \in \phi_{1,z}$, and therefore $t \in S_z$. (ii) Let $t \in \bigcup_{1 \trianglelefteq z \trianglelefteq y/x} S_z$. Then $t \in S_z$, where $z \in \mathbb{Z}_+$ is such that $z \trianglelefteq y/x$. Thus $t = u((y/x)/z)$, where $u \in \phi_{1,z}$. Since $u \in \phi_{1,z}$, it follows that $u \leq z$. Thus

$$\begin{aligned} & u \leq z \\ \Leftrightarrow\quad & u(y/x) \leq z(y/x) \\ \Leftrightarrow\quad & u((y/x)/z)z \leq z(y/x) \\ \Leftrightarrow\quad & u(y/x)/z \leq y/x \\ \Leftrightarrow\quad & t \leq y/x, \end{aligned}$$

and therefore $t \in \{s \mid s \in \mathbb{Z}_+,\, s \leq y/x\}$. Thus by (i) and (ii)

$$\{s \mid s \in \mathbb{Z}_+,\, s \leq y/x\} = \bigcup_{1 \trianglelefteq z \trianglelefteq y/x} S_z.$$

Let $z, w \in \mathbb{Z}_+$ be such that $z \trianglelefteq y/x$, $w \trianglelefteq y/x$, and $z \neq w$. Let us assume that $S_z \cap S_w \neq \emptyset$, and let $t \in S_z \cap S_w$. Then $t = u((y/x)/z)$, where $u \in \phi_{1,z}$, and $t = v((y/x)/w)$, where $v \in \phi_{1,w}$. Thus $u((y/x)/z) = v((y/x)/w)$, and therefore $u((y/x)/z)zw = v((y/x)/w)zw$. Thus $uw = zv$, and therefore $z \trianglelefteq uw$ and $w \trianglelefteq vz$. Since $\mathrm{gcf}(u, z) = 1$ and $\mathrm{gcf}(v, w) = 1$, it follows that $z \trianglelefteq w$ and $w \trianglelefteq z$, and therefore $z = w$. This contradicts the fact that $z \neq w$, and therefore $S_z \cap S_w = \emptyset$. Thus

$$\forall\, z, w \in \mathbb{Z}_+ : (z \trianglelefteq y/x,\, w \trianglelefteq y/x,\, z \neq w) \Rightarrow S_z \cap S_w = \emptyset.$$

Since $\#S_z = \#\phi_{1,z}$ and $S_z \cap S_w = \emptyset$ if $z \neq w$, it follows that

$$\#\{s \mid s \in \mathbb{Z}_+,\, s \leq y/x\} = \#\Big(\bigcup_{1 \trianglelefteq z \trianglelefteq y/x} S_z\Big) = \sum_{1 \trianglelefteq z \trianglelefteq y/x} \#\phi_{1,z},$$

and therefore

$$y/x = \sum_{1 \trianglelefteq z \trianglelefteq y/x} \phi(1,z).$$

Thus by Lemma 5.4

$$\zeta_1(1, y/x) = y/x = \sum_{1 \trianglelefteq z \trianglelefteq y/x} \phi(1,z) = \sum_{\substack{1 \trianglelefteq z \trianglelefteq y/x \\ zw = 1\cdot(y/x)}} \phi(1,z)\zeta(1,w) = (\phi * \zeta)(1, y/x).$$

Since by Theorems 9.7 and 6.16 $\phi \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ and $\zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ are completely translation invariant, respectively, it follows by Theorem 6.23 that $\phi * \zeta$ is completely translation invariant. Since by Theorem 9.4 $\zeta_1 \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is also completely translation invariant, it follows by Theorem 6.18 that $\zeta_1(x,y) = (\phi * \zeta)(x,y)$. Thus $\zeta_1 = \phi * \zeta$, and therefore by Lemma 5.4.

$$y/x = \zeta_1(x,y) = (\phi * \zeta)(x,y) = \sum_{\substack{x \trianglelefteq z \trianglelefteq y \\ zw = xy}} \phi(x,z)\zeta(x,w) = \sum_{x \trianglelefteq z \trianglelefteq y} \phi(x,z).$$

□

Theorem 9.8. *The power function $\zeta_1 \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is the divisibility order summatory function of the Euler function $\phi \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, i.e. $\zeta_1 = \phi * \zeta$.*

Proof. Follows by Lemma 9.2 and Theorem 5.15. □

Theorem 9.9. *The Euler function $\phi \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is factorable.*

Proof. By Theorems 9.6 and 8.28 $\zeta_1 \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is factorable, and therefore by Theorems 9.8 and 8.14 $\phi \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is factorable. □

Theorem 9.10. *The Euler function $\phi \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ satisfies the following formula:*

$$\phi = \zeta_1 * \mu, \quad \text{where } \zeta_1, \mu \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq].$$

Proof. Follows by Theorems 9.8 and 5.31. □

Theorem 9.11. *The following holds for the Euler function $\phi \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$:*

$$\forall\, x \in \mathbb{Z}_+ : \forall\, p \in \mathbb{P} : \forall\, m, n \in \mathbb{N} : \phi(xp^n, xp^{n+m+1}) = p^{m+1} - p^m.$$

Proof. Let $x \in \mathbb{Z}_+$, $p \in \mathbb{P}$, and $m, n \in \mathbb{N}$. Then by Theorems 9.7 and 9.10, Lemma 5.13, and Theorem 5.30

$$\begin{aligned}
\phi(xp^n, xp^{n+m+1}) &= \phi(p^n, p^{n+m+1}) \\
&= (\zeta_1 * \mu)(p^n, p^{n+m+1}) \\
&= \sum_{\substack{p^n \trianglelefteq z \trianglelefteq p^{n+m+1} \\ zw = p^n p^{n+m+1}}} \zeta_1(p^n, z)\mu(p^n, w) \\
&= \zeta_1(p^n, p^{n+m})\mu(p^n, p^{n+1}) + \zeta_1(p^n, p^{n+m+1})\mu(p^n, p^n) \\
&= \zeta_1(p^n, p^{n+m+1}) - \zeta_1(p^n, p^{n+m}) \\
&= p^{n+m+1}/p^n - p^{n+m}/p^n \\
&= p^{m+1} - p^m.
\end{aligned}$$

□

Theorem 9.12. *The following holds for the Euler function $\phi \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$:*

$$\forall\, x, y \in \mathbb{Z}_+ : x \trianglelefteq y \Rightarrow \phi(x, y) = y/x \prod_{\substack{p \trianglelefteq y/x \\ p \in \mathbb{P}}} (1 - p^{-1}).$$

Proof. Let $x, y \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$. Then $y = xz$, where $z \in \mathbb{Z}_+$, and therefore $z = y/x$. Let us note that $y(p) = x(p) + z(p)$ for all $p \in \mathbb{P}$, where $x = \prod_{p \in \mathbb{P}} p^{x(p)}$, $y = \prod_{p \in \mathbb{P}} p^{y(p)}$, and $z = \prod_{p \in \mathbb{P}} p^{z(p)}$, and therefore by Theorems 9.9, 8.9, and 9.11

$$\begin{aligned}
\phi(x, y) &= \prod_{p \in \mathbb{P}} \phi(p^{x(p)}, p^{y(p)}) \\
&= \prod_{x(p) < y(p)} \phi(p^{x(p)}, p^{y(p)}) \\
&= \prod_{z(p) \neq 0} \phi(p^{x(p)}, p^{x(p)+z(p)}) \\
&= \prod_{z(p) \neq 0} \left[p^{z(p)} - p^{z(p)-1}\right] \\
&= \prod_{z(p) \neq 0} p^{z(p)}(1 - p^{-1}) \\
&= \Big[\prod_{z(p) \neq 0} p^{z(p)}\Big]\Big[\prod_{z(p) \neq 0} (1 - p^{-1})\Big] \\
&= \Big[\prod_{p \in \mathbb{P}} p^{z(p)}\Big]\Big[\prod_{\substack{p \trianglelefteq z \\ p \in \mathbb{P}}} (1 - p^{-1})\Big] \\
&= y/x \prod_{\substack{p \trianglelefteq y/x \\ p \in \mathbb{P}}} (1 - p^{-1}).
\end{aligned}$$

□

The C-convolution of the Euler function and zeta function, i.e. $\phi *_C \zeta$, where $\phi, \zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, is C-discriminative.

Theorem 9.13. *The C-convolution $\phi *_C \zeta$, where $\phi, \zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, is C-discriminative.*

Proof. Let $x, y \in \mathbb{Z}_+$ be such that $\omega(x, xy) = n > 1$. Let $u, v \in \mathbb{Z}_+$ be such that $y = uv$, where

$$y = \prod_{p \in \mathbb{P}} p^{y(p)} = \prod_{i=1}^{n} p_i^{y(p_i)}, \quad u = \prod_{i=1}^{n-1} p_i^{y(p_i)}, \quad v = p_n^{y(p_n)}.$$

Then $\mathrm{gcf}(xu, xv) = x$, $\mathrm{lcm}(xu, xv) = xy$, $xu \neq x$, $xv \neq xy$, $xu \neq xy$, and $xv \neq x$. Thus by Lemmas 5.15 and 5.3

$$\begin{aligned}(\phi *_C \zeta)(x, xy) &= \sum_{\substack{\mathrm{gcf}(z,w)=x \\ \mathrm{lcm}(z,w)=xy}} \phi(x, z)\zeta(x, w) \\ &= \phi(x, x)\zeta(x, xy) + \phi(x, xy)\zeta(x, x) + \sum_{\substack{\mathrm{gcf}(z,w)=x \\ \mathrm{lcm}(z,w)=xy \\ z\neq x,\, w\neq xy \\ z\neq xy,\, w\neq x}} \phi(x, z)\zeta(x, w),\end{aligned}$$

where (by Lemmas 5.15 and 9.1)

$$\sum_{\substack{\mathrm{gcf}(z,w)=x \\ \mathrm{lcm}(z,w)=xy \\ z\neq x,\, w\neq xy \\ z\neq xy,\, w\neq x}} \phi(x, z)\zeta(x, w) > 0,$$

and therefore $(\phi *_C \zeta)(x, xy) \neq \phi(x, x)\zeta(x, xy) + \phi(x, xy)\zeta(x, x)$. Thus by the contraposition principle

$$\begin{aligned}\forall\, x, y \in \mathbb{Z}_+ : (\phi *_C \zeta)(x, xy) = \phi(x, x)\zeta(x, xy) + \phi(x, xy)\zeta(x, x) \\ \Rightarrow \omega(x, xy) \leq 1.\end{aligned}$$

and therefore by Definition 7.2 $\phi *_C \zeta$ is C-discriminative. □

The following theorem presents a characterization of a semifactorable arithmetic incidence function.

Theorem 9.14. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is semifactorable if and only if*

(i) $\forall\, x \in \mathbb{Z}_+ : f(x, x) = 1$,

(ii) $f(\phi *_C \zeta) = (f\phi) *_C f$.

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. Let us first assume that f is semifactorable. Then by Theorem 7.15 (i) and (ii) hold.

Let us next assume that (i) and (ii) hold. Since by Theorem 9.13 $\phi *_C \zeta$ is C-discriminative, it follows by Theorem 7.17 that f is semifactorable. □

The D-convolution of the Euler function and zeta function, i.e. $\phi * \zeta$, where $\phi, \zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, is D-discriminative.

Theorem 9.15. *The D-convolution $\phi * \zeta$, where $\phi, \zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, is D-discriminative.*

Proof. Let $x, y \in \mathbb{Z}_+$, where $y \neq 1$ and $y \notin \mathbb{P}$. Let $u, v \in \mathbb{Z}_+$ be such that $1 < u < y$, $1 < v < y$, and $y = uv$. Then $x \lhd xu \lhd xy$ and $xu \cdot xv = x \cdot xy$. Thus by Lemma 5.3

$$\begin{aligned}(\phi * \zeta)(x, xy) &= \sum_{\substack{x \trianglelefteq z \trianglelefteq xy \\ zw = x \cdot xy}} \phi(x, z)\zeta(x, w) \\ &= \phi(x, x)\zeta(x, xy) + \phi(x, xy)\zeta(x, x) + \sum_{\substack{x \lhd z \lhd xy \\ zw = x \cdot xy}} \phi(x, z)\zeta(x, w),\end{aligned}$$

where (by Lemma 9.1)

$$\sum_{\substack{x \lhd z \lhd xy \\ zw = x \cdot xy}} \phi(x, z)\zeta(x, w) > 0,$$

and therefore $(\phi * \zeta)(x, xy) \neq \phi(x, x)\zeta(x, xy) + \phi(x, xy)\zeta(x, x)$. Thus by the contraposition principle

$$\begin{aligned}&\forall\, x, y \in \mathbb{Z}_+ : \\ &\qquad (\phi * \zeta)(x, xy) = \phi(x, x)\zeta(x, xy) + \phi(x, xy)\zeta(x, x) \Rightarrow (y = 1 \text{ or } y \in \mathbb{P}),\end{aligned}$$

and therefore by Definition 7.8 $\phi * \zeta$ is D-discriminative. □

The following theorem presents a characterization of a completely semifactorable arithmetic incidence function.

Theorem 9.16. *A function $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is completely semifactorable if and only if*

(i) $\forall\, x \in \mathbb{Z}_+ : f(x, x) = 1$,

(ii) $f\zeta_1 = (f\phi) * f$.

Proof. Let $f \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$. Let us first assume that f is completely semifactorable. Let us note that by Theorems 9.8 and 5.9 (ii) is equivalent to $f(\phi * \zeta) = (f\phi) * (f\zeta)$. Thus by Theorem 7.40 (i) and (ii) hold.

Let us next assume that (i) and (ii) hold. Since by Theorem 9.15 $\phi * \zeta$ is D-discriminative and by Theorems 9.8 and 5.9 $f(\phi * \zeta) = (f\phi) * (f\zeta)$, it follows by Theorem 7.42 that f is completely semifactorable. □

9.4 Factor Functions

The family of *factor functions* of the factor lattice $(\mathbb{Z}_+, \trianglelefteq)$ is a generalization of the corresponding arithmetic functions.

Definition 9.5. Let $\alpha \in \mathbb{N}$. The *factor function* $\sigma_\alpha \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is defined as follows:

$$\sigma_\alpha : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C} : \sigma_\alpha(x, y) = \begin{cases} \sum\limits_{x \trianglelefteq z \trianglelefteq y} (z/x)^\alpha, & \text{if } x \trianglelefteq y; \\ 0, & \text{otherwise.} \end{cases}$$

The letter τ is used to denote the factor function $\sigma_0 \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, and therefore

$$\tau : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C} : \tau(x, y) = \sum_{x \trianglelefteq z \trianglelefteq y} 1,$$

i.e. the value of $\tau(x, y)$ is the number of elements of the interval $[\,x, y\,]$. Consequently, the factor function τ is referred to as the *cardinality function*.

The letter σ is used to denote the factor function $\sigma_1 \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, and therefore

$$\sigma : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C} : \sigma(x, y) = \sum_{x \trianglelefteq z \trianglelefteq y} z/x,$$

i.e. the value of $\sigma(x, y)$ is the sum of positive factors of y/x. Consequently, the factor function σ is referred to as the *sum of factors of quotient function*.

Theorem 9.17. *The factor function $\sigma_\alpha \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ is the divisibility order summatory function of the power function $\zeta_\alpha \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$.*

Proof. Follows by Definitions 9.5, 9.2, and 5.18. □

Theorem 9.18. *The factor function $\sigma_\alpha \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ satisfies the following formula:*

$$\sigma_\alpha = \zeta_\alpha * \zeta, \quad \textit{where } \zeta_\alpha, \zeta \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq].$$

Proof. Follows by Theorems 9.17 and 5.15. □

For example, $\tau = \zeta * \zeta$ (see Theorems 7.41 and 7.43).

Theorem 9.19. *The factor functions $\sigma_\alpha \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ are completely translation invariant.*

Proof. By Theorem 9.4 $\zeta_\alpha \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ are completely translation invariant, and therefore by Theorems 9.17 and 6.26 $\sigma_\alpha \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ are completely translation invariant. □

Theorem 9.20. *The factor functions $\sigma_\alpha \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ are factorable.*

Proof. By Theorems 9.6 and 8.28 $\zeta_\alpha \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ are factorable, and therefore by Theorems 9.17 and 8.14 $\sigma_\alpha \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$ are factorable. □

Theorem 9.21. *The following hold for the cardinality function $\tau \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$:*

(i) $\forall\, x \in \mathbb{Z}_+ : \forall\, p \in \mathbb{P} : \forall\, m, n \in \mathbb{N} : \tau(xp^n, xp^{n+m}) = m + 1,$

(ii) $\forall\, x, y \in \mathbb{Z}_+ : x \trianglelefteq y \Rightarrow \tau(x, y) = \prod_{p\in\mathbb{P}}[y(p) - x(p) + 1],$
where $x = \prod_{p\in\mathbb{P}} p^{x(p)}$ *and* $y = \prod_{p\in\mathbb{P}} p^{y(p)}$.

Proof. (i) Let $x \in \mathbb{Z}_+$, $p \in \mathbb{P}$, and $m, n \in \mathbb{N}$. Then by Theorems 9.19 and 6.18, Lemma 5.13, and the sum of arithmetic sequence

$$\tau(xp^n, xp^{n+m}) = \tau(1, p^m) = \sum_{1 \trianglelefteq z \trianglelefteq p^m} 1 = \sum_{k=0}^{m} 1 = m + 1.$$

(ii) Let $x, y \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$, where $x = \prod_{p\in\mathbb{P}} p^{x(p)}$ and $y = \prod_{p\in\mathbb{P}} p^{y(p)}$. Then by Theorems 9.20 and 8.9 and (i)

$$\tau(x, y) = \prod_{p\in\mathbb{P}} \tau(p^{x(p)}, p^{y(p)}) = \prod_{p\in\mathbb{P}} \tau(p^{x(p)}, p^{x(p)+y(p)-x(p)}) = \prod_{p\in\mathbb{P}} (y(p) - x(p) + 1).$$

□

Theorem 9.22. *The following hold for the factor functions $\sigma_\alpha \in \mathbb{I}[\mathbb{Z}_+, \trianglelefteq]$, where $\alpha \neq 0$:*

(i) $\forall\, x \in \mathbb{Z}_+ : \forall\, p \in \mathbb{P} : \forall\, m, n \in \mathbb{N} : \sigma_\alpha(xp^n, xp^{n+m}) = \frac{p^{\alpha(m+1)}-1}{p^\alpha - 1},$

(ii) $\forall\, x, y \in \mathbb{Z}_+ : x \trianglelefteq y \Rightarrow \sigma_\alpha(x, y) = \prod_{p\in\mathbb{P}} \frac{p^{\alpha(y(p)-x(p)+1)}-1}{p^\alpha - 1},$
where $x = \prod_{p\in\mathbb{P}} p^{x(p)}$ *and* $y = \prod_{p\in\mathbb{P}} p^{y(p)}$.

Proof. (i) Let $\alpha \in \mathbb{N}$ be such that $\alpha \neq 0$, and let $x \in \mathbb{Z}_+$, $p \in \mathbb{P}$, and $m, n \in \mathbb{N}$. Then by Theorems 9.19 and 6.18, Lemma 5.13, and the sum of geometric sequence

$$\sigma_\alpha(xp^n, xp^{n+m}) = \sigma_\alpha(1, p^m) = \sum_{1 \trianglelefteq z \trianglelefteq p^m} z^\alpha = \sum_{k=0}^{m} (p^k)^\alpha = \sum_{k=0}^{m} (p^\alpha)^k = \frac{p^{\alpha(m+1)} - 1}{p^\alpha - 1}.$$

(ii) Let $\alpha \in \mathbb{N}$ be such that $\alpha \neq 0$, and let $x, y \in \mathbb{Z}_+$ be such that $x \trianglelefteq y$, where $x = \prod_{p\in\mathbb{P}} p^{x(p)}$ and $y = \prod_{p\in\mathbb{P}} p^{y(p)}$. Then by Theorems 9.20 and 8.9 and (i)

$$\begin{aligned}
\sigma_\alpha(x, y) &= \prod_{p\in\mathbb{P}} \sigma_\alpha(p^{x(p)}, p^{y(p)}) \\
&= \prod_{p\in\mathbb{P}} \sigma_\alpha(p^{x(p)}, p^{x(p)+y(p)-x(p)}) \\
&= \prod_{p\in\mathbb{P}} \frac{p^{\alpha(y(p)-x(p)+1)} - 1}{p^\alpha - 1}.
\end{aligned}$$

□

Remark. Theorems 9.21 and 9.22 generalize the corresponding results established for arithmetic functions (see e.g. [3, pp. 38–39], [33, pp. 234–235]).

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